Fisher Information Universally Identifies Quantum Resources

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We show that both the classical as well as the quantum definitions of the Fisher information faithfully identify resourceful quantum states in general quantum resource theories, in the sense that they can always distinguish between states with and without a given resource. This shows that all quantum resources confer an advantage in metrology, and establishes the Fisher information as a universal tool to probe the resourcefulness of quantum states. We provide bounds on the extent of this advantage, as well as a simple criterion to test whether different resources are useful for the estimation of unitarily encoded parameters. Finally, we extend the results to show that the Fisher information is also able to identify the dynamical resourcefulness of quantum operations.

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Introduction.—The Fisher information (FI) plays an important foundational role in quantum information science. In quantum metrology and sensing, it determines the ultimate limits of precision of our measurement devices via the well-known quantum Cramér-Rao bound [1–3]. Existing applications include interferometry [4–6], magnetometry [7,8], thermometry [9,10], quantum illumination [11–13], displacement sensing [14,15], among others [16]. Crucially, such applications exploit the use of well-studied nonclassical quantum properties such as coherence [17,18], entanglement [19,20], and negative quasiprobabilities [21,22] in order to demonstrate the intrinsic superiority of quantum measurement devices over classical ones. FI has also been used to study nonclassical features of quantum systems such as quantum coherence [23,24] and entanglement [25,26].

Traditionally, different notions of nonclassicality in quantum mechanics have been studied independently. As such, the theoretical tools and quantities that were developed in the past typically probe a single nonclassical feature at a time. However, recent developments have made tremendous strides in providing a unified framework to study not only several disparate notions of nonclassicality [27–29], but also more general resources of quantum systems [30,31]. This has led to the discovery of physical tasks and operational quantities that are relevant in not just one particular resource theory, but also in general settings. This motivates us to consider quantities that are universally applicable in the sense that they maintain a physically meaningful interpretation while being able to identify *every* state which is considered “nonclassical” or “resourceful” within the physical constraints of the given resource theory [32–40]. An example of such quantities would be the class of robustness measures [41], which are well-defined quantifiers of any quantum resource that find use in quantifying the operational advantages of resources in channel discrimination tasks [35,37,42]. Not all meaningful quantities can identify all resource states of interest—for instance, in the theory of quantum entanglement, there exist important tasks such as distillation or quantum teleportation for which certain classes of entangled states are useless [19,43], and so quantities based on such tasks fail to faithfully characterize entanglement as a resource. This raises the questions of which tasks and quantities can be considered as universal witnesses of general quantum resources, and in which settings different notions of quantumness can provide tangible practical advantages.

In this work, we show that the FI universally characterizes the resources of quantum states, regardless of the specific resource in consideration, in that it is able to identify every resourceful state in general quantum resource theories. An immediate implication of this result is that quantum resources are always useful for quantum metrology, since there always exists some metrological problem where a resourceful probe outperforms a probe which does not possess a given resource. This also implies that the FI can be used as a generic tool to probe the resources of any system. We then establish theoretical bounds on the advantages provided by a given quantum states by relating the FI to the robustness measure of the given resource. We additionally provide a simple criterion for testing whether a given resource is useful for unitary parameter estimation. Finally, we also show that the FI can identify the resources of quantum operations in general resource theories.

Preliminaries.—We first define what a general quantum resource theory means in our context. Let $\mathcal{S}$ be the state space that describes a quantum system. A quantum resource theory is composed of a well-defined set of free quantum states $\mathcal{F}$—depending on the setting, these can be understood as the classical states, or the states without the given resource. Such states are accompanied by some set of free quantum operations $\mathcal{O}$. In order to account for as many possible formulations of quantum resources as possible, only a minimal set of assumptions are imposed on the sets $\mathcal{F}$ and $\mathcal{O}$. The set of free states $\mathcal{F}$ is assumed to be some closed and convex, but otherwise arbitrary subset of the state space $\mathcal{S}$. The physical interpretation of the assumption of convexity is that if one statistically mixes two free states together, the output will remain free. Given a well-defined set $\mathcal{F}$, we assume that $\mathcal{O}$ is some set of quantum maps that satisfies $\Phi(\sigma) \in \mathcal{F}$ if $\sigma \in \mathcal{F}$ and $\Phi \in \mathcal{O}$. Note that not all operations satisfying $\Phi(\sigma) \in \mathcal{F}$ are necessarily operational—in practice, the given choice of $\mathcal{O}$ is typically motivated by physical considerations. However, we will not need to assume any specific features of the free operations, allowing the very minimal assumptions made on $\mathcal{F}$ and $\mathcal{O}$ to maximize the generality of the subsequent results.

We now define the FI. In a typical scenario, there are two types of FI considered in quantum information. The first type is the FI one obtains from the classical post-processing of statistical data. This data is typically obtained from the measurement output of a fixed measurement setup. We will refer to this as the classical FI (CFI), and denote it as F_C.

The second type of FI is the maximum CFI one can obtain over all possible quantum measurements. This is typically called the quantum FI (QFI), and will here be denoted F_Q. By definition, we see that the CFI is a simple lower bound to the QFI.

Suppose we have a quantum channel Φ_θ that depends on some real parameter θ, and we would like to estimate θ. In order to do this, we pass a state ρ, called a probe, through the quantum channel Φ_θ and then perform a quantum measurement (positive operator-valued measure) $M = \{M_i\}$, where M_i are positive operators such that $\sum_i M_i = \mathbb{1}$. This results in the measurement statistics $P(i|\theta) = \mathrm{Tr}[\Phi_\theta(\rho) M_i]$. The maximum information about θ that we can obtain from the statistics $P(i|\theta)$ is quantified by the CFI, which is given by [44,45]:

$$F_C(\rho|\Phi_\theta, M) \coloneqq \sum_i P(i|\theta) \left[\frac{\partial \log P(i|\theta)}{\partial \theta}\right]^2. \quad (1)$$

In most cases, the quantum channel is fixed, so we can suppress the dependence on Φ_θ and use the simplified notation $F_C(\rho|M)$ instead. A similar notation will also be employed for the QFI F_Q.

In general, the FI is to be evaluated with respect to some given value of θ. Since the QFI $F_Q(\rho)$ is just the CFI maximized over all possible measurements M, the former depends only on the state ρ and the quantum channel Φ_θ, and we have $F_C(\rho|M) \leq F_Q(\rho)$. Formally, the QFI is given by the expression

$$F_Q(\rho) \coloneqq \mathrm{Tr}(\rho_\theta D_\theta^2), \quad (2)$$

where D_θ is the symmetric logarithmic derivative [46,47], satisfying the equation $\partial/\partial\theta \rho_\theta = \{\rho_\theta, D_\theta\}/2$, and where $\rho_\theta = \Phi_\theta(\rho)$.

Resourcefulness from the FI.—We will now establish our main result, which is that the FI can reveal general quantum resources. In order to do this, we need to demonstrate that for any resourceful state $\rho \notin \mathcal{F}$, there always exists a metrological problem represented by some quantum channel Φ_θ together with some measurement M, where the resulting FI correctly identifies ρ to possess some resources.

Suppose we would like to witness the resources of a state via the CFI. One way to go about doing this is to consider the quantity

$$N_C(\rho|M) \coloneqq F_C(\rho|M) - \max_{\sigma \in \mathcal{F}} F_C(\sigma|M), \quad (3)$$

where $F_C(\rho|M)$ is the CFI obtained by performing the measurement M on the state $\Phi_\theta(\rho)$, and the maximization is over $\mathcal{F}$, the set of free states in any given resource theory. N_C then quantifies the minimum quantum advantage of a resourceful state ρ over all possible classical states for a given metrological problem.

We can also consider a similar definition using the QFI:

$$N_Q(\rho) \coloneqq F_Q(\rho) - \max_{\sigma \in \mathcal{F}} F_Q(\sigma). \quad (4)$$

We see that if $N_Q(\rho) > 0$ or $N_C(\rho|M) > 0$, then the FI that is obtained using ρ exceeds that which can be obtained using any resourceless state $\sigma \in \mathcal{F}$. Since the excess FI can only be attributed to the given resource, the state ρ must be resourceful.

Upon first inspection, one may expect, since the QFI contains more information about the metrological utility of the quantum state than the CFI, that N_Q performs better than N_C at identifying the resourcefulness of states. This is in fact incorrect. To see this, recall that the QFI is the CFI optimized over all possible quantum measurements. Suppose $M^\star$ is the optimal measurement. This implies that for any channel Φ_θ, it is always possible to find a measurement $M^\star$ such that $N_C(\rho|M^\star) \geq N_Q(\rho)$. In general, we therefore see that the gap between resourceful and free states is larger using the CFI compared to the QFI, i.e., the CFI is able to identify more states as resourceful. Indeed, there exist scenarios where N_Q witnesses strictly fewer states than N_C. This is further discussed in the Supplemental Material [48].

In the following Theorem, we show that both the classical and the quantum versions of the FI can be used to identify general quantum resources.

Theorem 1. There exists a parameter estimation problem with quantum channel Φ_θ and measurement M that satisfies $N_C(\rho|M) > 0$ and $N_Q(\rho) > 0$ if and only if $\rho \notin \mathcal{F}$.

Proof.—We will first prove the statement for N_C. It is immediately clear that if $N_C(\rho|M) > 0$ or $N_Q(\rho) > 0$ for any parameter estimation problem, then ρ must be resourceful, which proves the "only if" direction.

To prove the converse direction, we use a result from Ref. [35], which states that if ρ is resourceful, then there exists a pair of quantum channels $\{A_0, A_1\}$ and POVM $\{\pi_0, \pi_1\}$ such that $p_{\mathrm{succ}}(\rho) > \max_{\sigma\in\mathcal{F}} p_{\mathrm{succ}}(\sigma)$, where $p_{\mathrm{succ}}(\rho) \coloneqq \frac{1}{2}\mathrm{Tr}[A_0(\rho)\pi_0] + \frac{1}{2}\mathrm{Tr}[A_1(\rho)\pi_1]$.

Let $\{A_0, A_1\}$ and POVMs $\{\pi_0, \pi_1\}$ be any such channel satisfying the above condition. We also introduce some state $\sigma_0 \neq \rho$ which will be specified later.

We now consider the following series of quantum channels acting on some arbitary state τ:

$$\Lambda_1(\tau) = \tau \otimes \frac{1}{2}\mathbb{1} \otimes [\theta|0\rangle\langle 0| + (1-\theta)|1\rangle\langle 1|], \quad (5)$$

$$\Lambda_2\circ\Lambda_1(\tau) = \tau \otimes \frac{1}{2}\mathbb{1} \otimes \theta|0\rangle\langle 0| + \sigma_0 \otimes \frac{1}{2}\mathbb{1} \otimes (1-\theta)|1\rangle\langle 1|, \quad (6)$$

$$\begin{aligned}\Lambda_3\circ\Lambda_2\circ\Lambda_1(\tau) = {} & A_0(\tau) \otimes \frac{1}{2}|0\rangle\langle 0| \otimes \theta|0\rangle\langle 0| \\ & + A_1(\tau) \otimes \frac{1}{2}|1\rangle\langle 1| \otimes \theta|0\rangle\langle 0| \\ & + A_0(\sigma_0) \otimes \frac{1}{2}|0\rangle\langle 0| \otimes (1-\theta)|1\rangle\langle 1| \\ & + A_1(\sigma_0) \otimes \frac{1}{2}|1\rangle\langle 1| \otimes (1-\theta)|1\rangle\langle 1|, \quad (7)\end{aligned}$$

$$\begin{aligned}&\Lambda_4\circ\Lambda_3\circ\Lambda_2\circ\Lambda_1(\tau) \\ &= \{\mathrm{Tr}[\Lambda_3\circ\Lambda_2\circ\Lambda_1(\tau)\pi_0 \otimes |0\rangle\langle 0| \otimes \mathbb{1}] \\ &\quad + \mathrm{Tr}[\Lambda_3\circ\Lambda_2\circ\Lambda_1(\tau)\pi_1 \otimes |1\rangle\langle 1| \otimes \mathbb{1}]\}|0\rangle\langle 0| \\ &\quad \{\mathrm{Tr}[\Lambda_3\circ\Lambda_2\circ\Lambda_1(\tau)\pi_0 \otimes |1\rangle\langle 1| \otimes \mathbb{1}] \\ &\quad + \mathrm{Tr}[\Lambda_3\circ\Lambda_2\circ\Lambda_1(\tau)\pi_1 \otimes |0\rangle\langle 0| \otimes \mathbb{1}]\}|1\rangle\langle 1|. \quad (8)\end{aligned}$$

Finally, we perform the projection $P_0 \coloneqq |0\rangle\langle 0|$ and $P_1 \coloneqq |1\rangle\langle 1|$ to obtain the statistics

$$P(0|\theta) = \theta p_{\mathrm{succ}}(\tau) + (1-\theta)p_{\mathrm{succ}}(\sigma_0) \quad (9)$$

and $P(1|\theta) = 1 - P(0|\theta)$.

Since $P(i|\theta)$, $i \in \{0,1\}$ is obtained from a series of quantum maps followed by a projection on τ, we see that this fits into the basic form $P(i|\theta) = \mathrm{Tr}[\Phi_\theta(\tau)M_i]$. Suppose we would like to estimate the parameter θ. We can then use Eq. (1) to evaluate the classical information of the statistics.

One may verify that

$$F_C(\tau|M) = \frac{(\frac{\partial P(0|\theta)}{\partial\theta})^2}{P(0|\theta)[1-P(0|\theta)]}. \quad (10)$$

Evaluating near the vicinity of $\theta = 0$, the resulting FI is

$$F_C(\tau|M) = \frac{[p_{\mathrm{succ}}(\tau) - p_{\mathrm{succ}}(\sigma_0)]^2}{p_{\mathrm{succ}}(\sigma_0)[1-p_{\mathrm{succ}}(\sigma_0)]}. \quad (11)$$

Recall that so far, the state σ_0 is not yet specified. We now choose it such that it satisfies $p_{\mathrm{succ}}(\sigma_0) = \min_{\sigma\in\mathcal{F}} p_{\mathrm{succ}}(\sigma)$. This means that the numerator is a monotonically increasing function of $p_{\mathrm{succ}}(\tau)$. We also see that the denominator does not depend on the state τ. Together with the fact that $p_{\mathrm{succ}}(\rho) > p_{\mathrm{succ}}(\sigma)$ for every $\sigma \in \mathcal{F}$, we must have $F_C(\rho|M) > \max_{\sigma\in\mathcal{F}} F_C(\sigma|M)$ and $N_C(\rho|M) > 0$ if ρ is resourceful. This shows the existence of at least one parameter estimation problem where $N_C(\rho|M) > 0$ if ρ contains some resource. For the special case where $p_{\mathrm{succ}}(\sigma_0) = 0$, then Eq. (10) becomes $F_C(\tau|M) = p_{\mathrm{succ}}(\tau)/\theta$ instead and a similar conclusion is reached. This is sufficient to prove both directions of the statement for N_C.

The equivalent statement for N_Q comes from the observation that the quantum map $\Lambda_4\circ\Lambda_3\circ\Lambda_2\circ\Lambda_1$ maps any input state to a diagonal state, and that the measurement M is also diagonal. We then use the fact that for diagonal states, the QFI is saturated by a measurement in the diagonal basis [1]. This is sufficient to show that $F_C(\rho|M) - \max_{\sigma\in\mathcal{F}} F_C(\sigma|M) = F_Q(\rho) - \max_{\sigma\in\mathcal{F}} F_Q(\sigma) = N_Q(\rho > 0)$, which proves the required statement. ■

Theorem 1 establishes that N_C and N_Q are both able to identify any resourceful state in general quantum resource theories. Both of the quantities therefore constitute faithful resource witnesses of direct physical relevance. This result also demonstrates the existence of a metrological advantage for any resourceful state. Theorem 1 can already be used to improve on previously known facts. For instance, Ref. [51] showed that a class of Werner states [52] which are entangled but admit a local hidden variable (LHV) model can exhibit metrological advantages over separable states. As we discuss in more detail in Ref. [48], our result not only shows that *any* entangled state, including LHV entangled Werner states and all bound entangled states, can provide such advantages, they can also lead to improved quantitative lower bounds on the extent of this advantage.

However, one can also be interested in understanding the metrological advantage precisely: *How much* advantage can be extracted from a given state ρ over all resourceless states? To this end, an even stronger statement can be proven which quantitatively relates the quantum advantage N_C and an important resource quantifier—the (generalized) robustness measure [41], which we denote $R(\rho)$.

So far, we have considered N_C given some fixed parameter estimation problem with encoding Φ_θ and measurement M. As a measure of the extent of the quantum advantage, it is reasonable to consider the maximum advantage one may obtain over all such encodings and measurements. Since the Fisher information can be scaled via a simple reparametrization $\theta \to k\theta$, we are also motivated to normalize the type of encoding channels over the set of free states. In light of these considerations, we can define the following quantity:

$$N_C^{\max}(\rho) \coloneqq \max_{\Phi_\theta\in\mathcal{P},M} N_C(\rho|\Phi_\theta, M), \quad (12)$$

where $\mathcal{P}$ is the set of all parameter estimation problems satisfying $\max_{\sigma\in\mathcal{F}} F_C(\sigma|\Phi_\theta, M) \le 1$. It can be shown that $N_C^{\max}$ is a resource monotone, upon which we elaborate further in the Supplemental Material [48]. Here we remark that other resource measures based on the FI have been employed in specific resource theories such as coherence and entanglement [24,53–55]. Reference [56] also considered the metrological gain of entangled states for unitary dynamics based on local Hamiltonians.

Theorem 2. In any resource theory, there exists a parameter estimation task task $\Phi_\theta \in \mathcal{P}$ that satisfies

$$R(\rho)^2 \le N_C(\rho|\Phi_\theta, M) \le R(\rho)^2 + 2R(\rho). \quad (13)$$

In particular, $N_C(\rho|\Phi_\theta, M) > 0$ iff $\rho \notin \mathcal{F}$, and $N_C^{\max}(\rho) \ge R(\rho)^2$ is a computable lower bound for any resource.

Theorem 2 provides a computable lower bound on the quantum advantage that can be extracted [48]. We stress that the robustness $R(\rho)$ can always be computed as a convex optimization problem. In many cases, such as the resource theories of coherence [57], multilevel coherence [58], and magic [59–62], it becomes an efficiently computable semidefinite program, while in many theories including entanglement [41,63,64] and multilevel entanglement [65] it can be computed analytically for all pure states. Furthermore, in the class of affine resource theories [66,67], which includes theories such as coherence and imaginarity [68,69], the lower bound of Theorem 2 is tight, in the sense that there always exists a task such that $N_C(\rho|\Phi_\theta, M) = R(\rho)^2$.

Taking this quantitative relationship further, it is natural to ask whether there is also an upper bound on the quantum advantage that a resource state affords in estimation tasks. Indeed, this is possible in the case where the decoding measurement has a binary outcome:

Theorem 3. For any parameter estimation task with encoding channel family Φ_θ and two-outcome measurement $M \equiv (P, \mathbb{1} - P)$, let $r \coloneqq F_C(\rho|M) < \infty$ and

$$\omega \coloneqq \left| \frac{\partial \mathrm{Tr} P \Phi_\theta(\rho)}{\partial \theta} \right|_{\theta=0}. \quad (14)$$

Then,

$$N_C(\rho|M) \le r - \frac{[R_S(\rho) + 1]^{-2}\omega^2}{\max_{\tau\in\mathcal{F}} \mathrm{Tr}[P\Phi_0(\tau)](1 - \mathrm{Tr}[P\Phi_0(\tau)])}$$
$$\le r - \frac{4\omega^2}{[R_S(\rho) + 1]^2}, \quad (15)$$

where $R_S(\rho)$ is the standard robustness of ρ with respect to $\mathcal{F}$ [41].

R_S is another operationally significant resource measure closely related to R, and like the latter, admits efficient SDP formulations in many resource theories. Thus, given any particular parameter estimation task with a two-outcome measurement, we have an efficiently computable upper bound to the quantum advantage N_C of any given resource state. The bound in the first line is less computationally feasible but tighter.

We note here that the parameter ω scales with the energy cost of applying the given family of encoding channels on ρ, with finer θ resolution costing more energy. We expand on this connection in Ref. [48], together with a proof of Theorem 3; there we also discuss why we suspect the upper bound (15) cannot be made independent of the estimation task in general. Finding upper bounds for the case of more general measurements is left for future work.

Quantum resources for unitary encodings.—We now consider the important special case where the quantum channel Φ_θ is a unitary encoding channel $\Phi_\theta(\rho) = U_\theta \rho U_\theta^\dagger$ and $U_\theta \coloneqq e^{-i\theta G}$. Here, G is some Hermitian operator specifying the unitary evolution, and is called the generator of the unitary encoding.

Given any unitary encoding generated by the Hermitian operator G, one may be interested to know whether the parameter estimation problem corresponding to G is benefited by having resourceful states in a given quantum resource theory. The following result establishes a simple criterion for determining whether G reveals the given resource.

Theorem 4. Consider any Hermitian generator G and convex set of free states $\mathcal{F}$. Let s^* be an optimal solution to the convex optimization problem

$$\underset{X\ge 0}{\text{maximize}} \quad 2\mathrm{Tr}[(G_A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes G_B)^2 X_{AB}]$$
$$\text{subject to} \quad \mathrm{Tr}_A X_{AB} = \mathrm{Tr}_B X_{AB} = \sigma \in \mathcal{F}.$$

Let $\lambda_{\max}, \lambda_{\min}$ denote the largest and smallest eigenvalues of the generator G, respectively. If $(\lambda_{\max} - \lambda_{\min})^2 > s^*$, then $N_Q(\rho) > 0$ for some ρ.

Proof.—One may immediately verify that the objective function $2\mathrm{Tr}[(G_A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes G_B)^2 X_{AB}]$ is linear and that the feasible set is convex, so s^* is the solution to a convex optimization problem.

To prove the statement, we just need to show that s^* upper bounds $\max_{\sigma\in\mathcal{F}} F_Q(\sigma)$. In general, for any generator G the maximum achievable QFI can be verified to be $(\lambda_{\max} - \lambda_{\min})^2$ [70], so if $(\lambda_{\max} - \lambda_{\min})^2 > s^* \ge \max_{\sigma\in\mathcal{F}} F_Q(\sigma)$, we necessarily have $N_Q(\rho) > 0$.

To see that s^* in indeed an upper bound, we use the fact that $F_Q(\sigma) \le 4\Delta_\sigma^2 G$, where $\Delta_\sigma^2 G$ is the variance of G given the state σ. We then observe that $X_{AB} = \sigma \otimes \sigma$ where $\sigma \in \mathcal{F}$ is a feasible solution. Finally, we observe that $2\mathrm{Tr}[(G_A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes G_B)^2 \sigma \otimes \sigma] = 4\Delta_\sigma^2 G$ so we must have $s^* \ge \max_{\sigma\in\mathcal{F}} F_Q(\sigma)$ as required. ∎

The convex optimization in Theorem 4 provides a direct method of testing of whether the particular parameter estimation problem generated by G will benefit from a given quantum resource. Note that this is a sufficient condition, so failure of the test does not necessarily imply the resource is not useful for this encoding. We also highlight that the criterion is based on the QFI-based quantity N_Q, but it also applies to the CFI case as well, since if $N_Q(\rho) > 0$, then we are guaranteed some measurement M for which $N_C(\rho|M) > N_Q(\rho) > 0$. A similar sufficient condition can be obtained for non unitary encoding channels, in terms of their unitary dilation.

We illustrate Theorem 4 with a simple worked example. Consider a qubit system, with the free set $\mathcal{F} = \{|0\rangle\langle 0|\}$ being a trivial set with only a single element. Let $G = \sigma_z$, the Pauli matrix in the z direction. In this case, the feasible set only has one state $X_{AB} = |0\rangle\langle 0| \otimes |0\rangle\langle 0|$, from which we can verify $s^* = 0$. Since $(\lambda_{\max} - \lambda_{\min})^2 = 4$, from Theorem 4, there must exist some state $\rho \notin \mathcal{F}$ such that $N_C(\rho) > 0$. One can repeat the same argument for the case where $\mathcal{F} = \{|1\rangle\langle 1|\}$. Since the FI is convex [2], ρ must outperform any convex mixture of $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, i.e., any incoherent quantum state. This is one way to verify that quantum coherence is a useful nonclassical resource for the unitary encoding generated by $G = \sigma_z$.

Identifying resourceful operations.—Thus far, we have considered the use of the FI to identify general resources in quantum states. Recall that every quantum resource theory is also accompanied by some set of free quantum operations $\mathcal{O}$. Just as any resourceful state can be defined as a state that is not in the set of free states $\mathcal{F}$, we can similarly define a resourceful operation as any quantum channel not in the set $\mathcal{O}$ [71,72]. It turns out that the FI can also universally distinguish operations with and without a given resource.

Theorem 5. For any set of free operations $\mathcal{O}$ and quantum map $\Xi \notin \mathcal{O}$, there exists a quantum trajectory ρ_θ on an extended Hilbert space such that the map $\mathbb{1} \otimes \Xi$ satisfies

$$F_C[\mathbb{1} \otimes \Xi(\rho_\theta)|M] > \max_{\Omega \in \mathcal{O}} F_C[\mathbb{1} \otimes \Omega(\rho_\theta)|M] \quad (16)$$

for some θ.

A full discussion of the proof can be found in Ref. [48]. Theorem 5 demonstrates that the FI plays a foundational role not just in the investigation of resourcefulness in states, but also in the study of the resources of quantum channels.

Conclusion.—Many quantities traditionally used to study the nonclassical features of quantum mechanics are typically relevant only when considering specific notions of nonclassicality. However, quantum advantages in different tasks rely on a broad range of quantum phenomena, motivating the study of physical quantities that can be used to identify *all* reasonable quantum resources. To this end, we showed that two Fisher information-based quantities N_C and N_Q, defined through the classical and quantum variants of the FI, respectively, are examples of such universally relevant operational quantities. This implies that every quantum resource can provide a quantum advantage in some parameter estimation problem. In this sense, any feasible notion of nonclassicality objectively always provides a quantum advantage in metrology, although it remains subjective as to whether such applications are relevant to the interests of an experimentalist. We also highlight that, while the focus of this work is on identifying (detecting) general quantum resources, it is also possible to construct resource measures—also called resource monotones—using N_C and N_Q. This is discussed in greater detail in the Supplemental Material [48].

We then provided a lower bound on the maximum extent of this quantum advantage in terms of the generalized robustness measure [35,37,41], which also universally identifies resourceful states. For the case of estimation problems with binary outcomes, we also provided an upper bound, in terms of a related quantity called the standard robustness. The special case of unitary encodings was also considered, where we provided a simple criterion to test whether a given quantum resource provides an advantage for a unitary encoding generated by a Hermitian operator G. Finally, we showed that not only does the FI universally identify resources of quantum states, it also universally witnesses resourceful quantum operations in every resource theory. These results solidify the central role that the FI plays in the study of quantum resources.

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