Computable Entanglement Cost under Positive Partial Transpose Operations

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(Received 17 July 2024; accepted 5 February 2025; published 7 March 2025)

Quantum information theory is plagued by the problem of regularizations, which require the evaluation of formidable asymptotic quantities. This makes it computationally intractable to gain a precise quantitative understanding of the ultimate efficiency of key operational tasks such as entanglement manipulation. Here, we consider the problem of computing the asymptotic entanglement cost of preparing noisy quantum states under quantum operations with positive partial transpose (PPT). By means of an analytical example, a previously claimed solution to this problem is shown to be incorrect. Building on a previous characterization of the PPT entanglement cost in terms of a regularized formula, we construct instead a hierarchy of semidefinite programs that bypasses the issue of regularization altogether, and converges to the true asymptotic value of the entanglement cost. Our main result establishes that this convergence happens exponentially fast, thus yielding an efficient algorithm that approximates the cost up to an additive error ε in time $\mathrm{poly}(D, \log(1/\varepsilon))$, where D is the underlying Hilbert space dimension. To our knowledge, this is the first time that an asymptotic entanglement measure is shown to be efficiently computable despite no closed-form formula being available.

DOI: 10.1103/PhysRevLett.134.090202

Introduction—Quantum Shannon theory studies the fundamental limitations on the manipulation of quantum information in the presence of external noise. Calculating those limits often involves computing certain functions that encapsulate the ultimate capabilities of information carriers. Paradigmatic examples include the various capacities of quantum channels, such as the classical [1,2], quantum [3–5], private [5,6], and entanglement-assisted [7,8] capacities, but also the operational entanglement measures that tell us how much entanglement can be extracted from a given bipartite quantum state, i.e., the distillable entanglement [9–13], and vice versa, how much entanglement must be invested to *create* that state [14–17]. This latter quantity, the entanglement cost, is the main focus of this work.

With the sole exception of the entanglement-assisted capacity, all of the above functions are expressed by regularized formulas, i.e., formulas that involve an explicit limit $n \to \infty$ over the number of uses of the channel or the available copies of the state. For example, by the Lloyd-Shor-Devetak theorem [3–5], the quantum capacity of a channel $\mathcal{N}$ equals $Q(\mathcal{N}) = \lim_{n\to\infty}(1/n)I_c(\mathcal{N}^{\otimes n})$, where $I_c(\mathcal{N})$ is the coherent information of $\mathcal{N}$, and $\mathcal{N}^{\otimes n}$ represents n parallel uses of $\mathcal{N}$. In stark contrast with classical information theory, for quantum channels it holds in general that $I_c(\mathcal{N}^{\otimes n}) \neq nI_c(\mathcal{N})$, meaning that evaluating the limit cannot be avoided. Such nonadditivity is a fundamental feature of most settings encountered in quantum information [18–23]. Analogously, the entanglement cost of preparing a state $\rho = \rho_{AB}$ using local operations and classical communication (LOCC) is given by

$$E_{c,\mathrm{LOCC}}(\rho) = \lim_{n\to\infty} \frac{1}{n} E_f\big(\rho^{\otimes n}\big), \tag{1}$$

where E_f is the entanglement of formation [11,24]. The precise nature of these formulas is not so important here; what is important, however, is that the regularization $n \to \infty$ makes them analytically hard to control and computationally intractable. Indeed, on the one hand the dimension of the quantum system on which $\mathcal{N}^{\otimes n}$ acts, or to which $\rho^{\otimes n}$ pertains, is exponential in n, quickly rendering numerical calculations infeasible as n grows; on the other, there is no guarantee of the quality of the approximation obtained by stopping at the nth level in any of these formulas—for

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instance, an unbounded n may be required to even check that the quantum capacity is nonzero [25]. The regularization thus appears to be an omnipresent curse that stifles almost every attempt to quantitatively understand the ultimate limitations of quantum information manipulation.

But is that really so? In this Letter we show how to overcome this fundamental obstacle in a specific case by efficiently calculating a type of entanglement cost—expressed as a regularized quantity—on all quantum states. To this end, we look at a problem that has been studied by many authors [26–30], but for which a full solution had not been found prior to our work: namely, zero-error asymptotic entanglement cost under "positive partial transpose" (PPT) operations. The roots of PPT transformations lie in the fundamental connection between entanglement and partial transposition identified by Peres [31] and the Horodeckis [32,33]. In the context of entanglement manipulation, PPT operations were introduced in the pioneering works of Rains [34,35] as a mathematically natural relaxation of the much more complicated set of LOCC operations, providing a convenient way to gain some insights into the latter and, importantly, giving hope for an easier understanding of some of the fundamental limits of entanglement transformations. Because of this, they attracted significant attention in the operational study of quantum entanglement, but even in this technically simpler setting the fundamental questions in entanglement manipulation remained unsolved.

By introducing an efficient algorithm for the evaluation of the asymptotic PPT entanglement cost, we solve this long-standing problem and exhibit an entanglement measure that is both *computable* and *operationally meaningful* for general—pure and mixed—quantum states. This can be contrasted with other entanglement measures, which are either defined in terms of operational tasks and thus suffer from the problem of regularization, making efficient evaluation impossible, or they are simply abstract mathematical constructions with no precise operational meaning. Prior to our work, the PPT entanglement cost was only known to be computable for a specific class of quantum states [26] (including pure states, Gaussian states, Werner states, and two-qubit states [36]), with its value given by a celebrated entanglement measure known as the logarithmic negativity [26,37,38]. The logarithmic negativity is one of the most widely employed entanglement measures, having found numerous applications in all corners of quantum physics [39–46], from quantum field theory [39] to condensed matter [42]. The success of the logarithmic negativity is due to two main reasons: first, its efficient computability [37], which makes it highly suitable for numerical applications, and second, its *partial* operational interpretation, which guarantees that it coincides with the zero-error PPT entanglement cost for this specific class of quantum states [26]. While encouraging, this partial operational interpretation is arguably not satisfactory if this measure is to be applied widely in quantum physics. Indeed, beyond this restricted class of states the negativity is not known to be an operational quantity, and it is unclear whether, e.g., the states considered in [40,41,46] genuinely belong to the class in question.

It was recently claimed that the PPT entanglement cost can be computed exactly for all states [27,28], but, as we show below (see also [47]), this claim relies on some erroneous assertions. In this work, building on the partial results of [27,28], we find the correct generalization of the logarithmic negativity that enjoys the operational interpretation of being the zero-error PPT entanglement cost for *all* quantum states. Crucially, our generalization is also efficiently computable, requiring almost the same computational time as the logarithmic negativity. This opens the door to numerous applications in quantum physics, serving as a fully operationally motivated alternative to the negativity as a computable entanglement quantifier.

Zero-error PPT entanglement cost—The goal of zero-error PPT entanglement dilution is to prepare n copies of a given bipartite quantum state $\rho = \rho_{AB}$ by consuming as few singlets Φ_2 (i.e., two-qubit maximally entangled states) as possible and using PPT operations only. Here, a channel Λ is called PPT if its partial transposition $\Gamma\circ\Lambda\circ\Gamma$ is also a valid quantum channel, where Γ is the partial transpose operation [31] defined as $\Gamma(X_A \otimes Y_B) = X_A \otimes Y_B^{\intercal}$. We say that a number R is an achievable rate if for all sufficiently large n there exists a PPT operation Λ_n with the property that $\Lambda_n(\Phi_2^{\otimes \lfloor Rn \rfloor}) = \rho^{\otimes n}$. By definition, the zero-error PPT entanglement cost of ρ, denoted as $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)$, is the infimum of all achievable rates R. In this work, we show how to compute $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)$ for any quantum state ρ.

In a nutshell, the key reason why many authors [26–30] have been interested in the problem of entanglement dilution under PPT operations is that it provides a more tractable model for the fundamental problem of entanglement dilution under LOCC operations, whose underlying figure of merit, given by (1), is computationally inaccessible in most cases of interest. In general, PPT operations are a superset of LOCC, but any PPT channel can be implemented in a stochastic manner by LOCC together with the assistance of PPT states [35,48]; since it is known that PPT states possess only a weak form of entanglement that cannot be distilled [49], such states can be considered as a "cheap" resource, thus providing intuition for the PPT operations being a prudent relaxation of the power of LOCC. Such PPT-based approaches have also attracted significant attention in other parts of entanglement theory [34,35,50,51].

Importantly, the optimal performance achievable under PPT operations also establishes bounds and no-go limits on what can be achieved under LOCC in practice. These bounds are often the tightest available, as evidenced in the contexts of channel capacities [52,53] and entanglement distillation [35]. Let us now consider the task of entanglement dilution under LOCC operations, which is defined as above but with LOCC operations replacing PPT operations. We denote the corresponding figure of merit by $E_{c,\mathrm{LOCC}}^{\mathrm{exact}}(\rho)$. Because of PPT being an outer approximation to LOCC, it holds that

$$E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) \leq E_{c,\mathrm{LOCC}}^{\mathrm{exact}}(\rho) \tag{2}$$

for all bipartite states $\rho = \rho_{AB}$. Our main result allows us to calculate the left-hand side, which is particularly significant because establishing computable lower bounds on $E_{c,\mathrm{LOCC}}^{\mathrm{exact}}(\rho)$ is *a priori* difficult, as to do that one needs to constrain *all* possible LOCC entanglement dilution protocols. (To prove an upper bound, on the contrary, it suffices to exhibit an explicit dilution protocol.)

We also remark that, in our definition, "zero error" means that we require the transformation of $\Phi_2^{\otimes \lfloor Rn \rfloor}$ into $\rho^{\otimes n}$ to be realized exactly. This is a model of entanglement dilution that has been studied before in several different contexts [16,26,54]. The setting contrasts with definitions that allow for an asymptotically vanishing error in the transformation [11,14,16]. However, in the Supplemental Material (SM) [55] we show that no substantial change occurs if we require that the error, instead of being exactly zero, decay to zero sufficiently fast—the relevant figure of merit is then still $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$.

Prior work—In a pioneering paper by Audenaert, Plenio, and Eisert [26], it was shown that the PPT entanglement cost $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)$ can be evaluated exactly for all bipartite states $\rho = \rho_{AB}$ that satisfy a condition known as "zero binegativity" [26,36,67]. Specifically, if $|\rho^\Gamma|^\Gamma \geq 0$, where X^Γ denotes the partial transpose $\Gamma(X)$ of an operator X and $|X| \coloneqq \sqrt{X^\dagger X}$ is its absolute value, then

$$E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) = \log_2 \|\rho^\Gamma\|_1 \eqqcolon E_N(\rho). \tag{3}$$

The expression on the right, E_N, is the logarithmic negativity [37,38], obtained by simply evaluating the trace norm $\|\cdot\|_1 \coloneqq \mathrm{Tr}|\cdot|$ of the partially transposed state. This framework thus provides a partial operational interpretation for E_N through its equality with the zero-error PPT cost $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ for some states. At the same time, Eq. (3) effectively solves the problem of computing $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ for states with zero binegativity because E_N is efficiently computable via a semidefinite program (SDP) [68–70],

$$E_N(\rho) = \log_2 \min\{\mathrm{Tr}S\colon -S \leq \rho^\Gamma \leq S\}. \tag{4}$$

But what to do for those states that have nonzero binegativity, i.e., satisfy $|\rho^\Gamma|^\Gamma \ngeq 0$? In their recent works [27,28,47], Wang and Wilde showed that $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ can be expressed as a regularization of another quantity called E_κ. They then made an even stronger claim that E_κ is in fact *additive*, meaning that $E_\kappa(\rho^{\otimes n}) \overset{?}{=} nE_\kappa(\rho)$ holds true for all bipartite states ρ and all positive integers n, thus completely eliminating the issue of regularization. This would imply a general computable solution for $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$: it would simply coincide with E_κ, which is computable via an SDP [27,28,47]. However, in Lemma 1 below we construct a simple counterexample that disproves the claimed additivity of E_κ. Its existence shows that in general $E_\kappa \neq E_{c,\mathrm{PPT}}^{\mathrm{exact}}$, thus invalidating the computable solution claimed in [27,28] and reopening the question of whether the asymptotic cost $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ can be efficiently evaluated. Further details concerning the claims of [27,28] can be found in [47].

The equivalence between the regularization of the quantity E_κ and the PPT entanglement cost revealed in [27,28] will still prove useful to us, albeit *a priori* it is not clear how it could lead to a computable formula for $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$—the daunting problem of regularization persists.

Main results—In this work, we completely solve the problem of computing the asymptotic zero-error PPT entanglement cost $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$. To do this, we construct a converging hierarchy of semidefinite programs that can be used to calculate $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ for any given state to any degree of precision *efficiently*, i.e., in a time polynomial in the underlying Hilbert space dimension and in $\log(1/\varepsilon)$, with ε being the additive error. The key quantities in our approach are a family of PPT entanglement monotones indexed by an integer $p \in \mathbb{N}$ and given by

$$E_{\chi,p}(\rho) \coloneqq \log_2 \chi_p(\rho), \tag{5}$$

where

$$\chi_p(\rho) \coloneqq \min\left\{\mathrm{Tr}S_p\colon -S_i \leq S_{i-1}^\Gamma \leq S_i, i = 0, \ldots, p, S_{-1} = \rho\right\} \tag{6}$$

is an SDP with variables $S_0, \ldots, S_p$. These quantities are increasing in p for every fixed ρ, and we refer to them as the "χ hierarchy." Note also that $E_{\chi,0} = E_N$, hence $E_{\chi,p}$ can be regarded as a generalization of the logarithmic negativity [37,38].

Importantly, we show that the χ hierarchy approximates the entanglement cost $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ from below, in the sense that $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) \geq E_{\chi,p}(\rho)$ for all states ρ and all p. The proof of this fact, which can be found in the SM [55], relies on the connection between the entanglement cost $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ and the regularized form of E_κ shown in [27,28]. Our first main result, the forthcoming Theorem 1, establishes that this approximation becomes increasingly tight as p increases, and the χ hierarchy gives the value of $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ *exactly* in the limit $p \to \infty$. This allows us to replace the limit in the number of copies n, which is what makes $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ difficult to compute, with a limit in the hierarchy level p. This already provides a "single-letter" formula for the PPT entanglement cost that no longer suffers from the curse of regularization.

However, because of the limiting procedure $p \to \infty$, it is still unclear if the expression can be evaluated easily. Crucially, in Theorem 1 we also show that calculating the limit of the χ hierarchy is indeed significantly easier than evaluating regularized expressions: the convergence to the true value of $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ is *exponentially fast* uniformly on all states, which opens the way to an accurate calculation of $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ in practice.

Theorem 1 (Exact expression of the cost)—For all bipartite states $\rho = \rho_{AB}$ on a system of minimal local dimension $d \coloneqq \min\{|A|, |B|\} \geq 2$, and all positive integers $p \in \mathbb{N}^+$, it holds that

$$0 \leq E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) - E_{\chi,p}(\rho) \leq \log_2 \frac{1}{1 - \left(1 - \frac{2}{d}\right)^p}, \quad (7)$$

entailing that

$$E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) = \lim_{p\to\infty} E_{\chi,p}(\rho). \quad (8)$$

The proof of Theorem 1 is outlined in the End Matter, with full details provided in the SM for interested readers [55]. Note that for every fixed value of d and for large p, the approximation error on the right-hand side of (7) can be estimated as $[1 - (2/d)]^p(\log_2 e)$. In other words, the speed of convergence in (8) is exponential in p and furthermore independent of ρ.

The single-letter formula (8) can be used to establish two notable properties of the zero-error PPT entanglement cost $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$, namely additivity and continuity [55, Secs. III.E, F]. Also, Theorem 1 yields immediately a simple solution in the qubit-qudit case ($d = 2$), generalizing Ishizaka's result that $E_{c,\mathrm{PPT}}^{\mathrm{exact}} = E_N$ for all two-qubit states [36].

Corollary 1—For all states $\rho = \rho_{AB}$ on a $2 \times n$ bipartite quantum system, it holds that

$$\begin{aligned} E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) &= E_{\chi,1}(\rho) \\ &= \log_2 \min\{\|S_0^\Gamma\|_1 :\ -S_0 \leq \rho^\Gamma \leq S_0\}. \end{aligned} \quad (9)$$

But the most important implication of Theorem 1 is that it allows us to construct an efficient algorithm that calculates $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ to any desired accuracy. This algorithm takes as input a bipartite state $\rho = \rho_{AB}$ and an error tolerance $\varepsilon > 0$, and returns as output a number $\tilde{E}$ such that $|E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) - \tilde{E}| \leq \varepsilon$. It works as follows: (1) use (7) to find p large enough so that $E_{\chi,p}(\rho)$ approximates $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)$ up to an error $\varepsilon/2$. A value of $p = \mathcal{O}(d \log(d/\varepsilon))$ is sufficient. (2) Solve the SDP in (6) up to an (additive) error $\varepsilon \ln(2)/2$. Taking the logarithm yields an estimate $\tilde{E}$ of $E_{\chi,p}(\rho)$ up to an (additive) error $\varepsilon/2$ [see (5)]. (3) Return $\tilde{E}$.

The time complexity of the above algorithm is analyzed in Theorem 2, which shows the core result of our work: the zero-error PPT entanglement cost can be efficiently computed. The key observation behind this result is that climbing the χ hierarchy up to level p introduces only polynomially many more constraints in the SDP in (5) and is thus relatively inexpensive.

Theorem 2 (Time to compute the cost)—Let ρ be a bipartite state on a system of total dimension D and minimal local dimension d. Then, the above algorithm computes $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)$ up to an additive error ε in time

$$\mathcal{O}\big((dD)^{6+o(1)} \mathrm{poly}\log(1/\varepsilon)\big). \quad (10)$$

The proof of Theorem 2 is sketched in the End Matter. Remarkably, Theorem 2 implies that the time complexity required to compute $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ is only marginally larger than that of computing the logarithmic negativity E_N, which is bounded by the time complexity required for the diagonalization of a $D \times D$ matrix—also polynomial in D and $\log(1/\epsilon)$. In short, while both $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ and E_N can be computed efficiently, $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$ stands out because, by definition, it has an operational meaning for all quantum states, unlike E_N.

Our result is the first of its kind for two distinct reasons. First, because it establishes the efficient computability of an operationally meaningful asymptotic entanglement measure (i.e., a distillable entanglement or an entanglement cost). There is no known algorithm to estimate any other such measure, not even under the simplifying zero-error assumption. Second, because efficient computability is shown without exhibiting a closed-form single-letter formula, but rather by describing a converging SDP hierarchy. To the extent of our knowledge, the only other case in quantum information theory where a similar situation arises is in [71, Theorem 5.1]. However, unlike ours, the algorithm described there is computationally extremely expensive, featuring an exponential dependence on d^3/ε. More generally, expressing difficult-to-compute quantities through converging SDP hierarchies is a technical tool that has found various uses in quantum information [53,72–76], but such applications typically do not result in efficiently computable algorithms or do not yield exact operational results.

Discussion and conclusions—In this Letter, we have provided a solution to the problem of efficiently calculating the zero-error PPT entanglement cost of arbitrary (finite-dimensional) quantum states. To the best of our knowledge, it is the first time that any operational asymptotic entanglement measure is shown to be efficiently computable. A particularly interesting feature of our construction is that it does not rely on a closed-form formula, but rather on a converging hierarchy of semidefinite programs that approximate the cost from above and below with controllable error.

Our solution identifies the correct generalization of the celebrated logarithmic negativity, which became popular in quantum physics for its efficient computability, despite being operationally meaningful only for states with zero binegativity. In contrast, our entanglement measure,

$\lim_{p\to\infty} E_{\chi,p}$, is not only efficiently computable—specifically, it can be efficiently computed via an algorithm with almost the same time complexity as the logarithmic negativity—but it also has operational meaning for *all* quantum states through its equality with the zero-error PPT entanglement cost $E^{\mathrm{exact}}_{c,\mathrm{PPT}}$. For states with zero binegativity, our entanglement measure coincides with the logarithmic negativity, but for other states it can be significantly smaller.

An open question in our analysis is whether the χ hierarchy collapses at any finite level for some—or even all—states. We found examples of states ρ such that $E_N(\rho) = E_{\chi,0}(\rho) < E_{\chi,1}(\rho)$ and also $E_{\chi,1}(\rho) < E_{\chi,2}(\rho)$, but we were not able to ascertain whether there exists in general a gap between $E_{\chi,2}$ and $E_{\chi,3}$. If $E_{\chi,2} = E_{\chi,3}$ holds in general, then this would mean that $E^{\mathrm{exact}}_{c,\mathrm{PPT}} = E_{\chi,2}$, and thus the cost could be computed with a simple single-letter formula. While this would be a considerable simplification from the analytical standpoint, we stress that it will only entail a $\mathrm{poly}(d)$ improvement in the time complexity of evaluating it numerically.

Note added—The issue with the original argument by Wang and Wilde that leads to the additivity violation for E_κ (Lemma 1) is discussed in detail in the erratum [47].

Acknowledgments—L. L. thanks Andreas Winter for enlightening correspondence on zero-error information theory, as well as the organizers and attendees of the workshop “Quantum information” (Les Diablerets, Switzerland, February 25 to March 1, 2024), where this work was first presented, for stimulating and entertaining discussions. L. L. and B. R. thank Xin Wang and Mark M. Wilde for feedback on the first draft of this Letter. L. L. is supported by MIUR (Ministero dell’Istruzione, dell’Università e della Ricerca) through the project “Dipartimenti di Eccellenza 2023–2027” of the “Classe di Scienze” department at the Scuola Normale Superiore.

[1] A. S. Holevo, The capacity of the quantum channel with general signal states, IEEE Trans. Inf. Theory **44**, 269 (1998).
[2] B. Schumacher and M. D. Westmoreland, Sending classical information via noisy quantum channels, Phys. Rev. A **56**, 131 (1997).
[3] S. Lloyd, Capacity of the noisy quantum channel, Phys. Rev. A **55**, 1613 (1997).
[4] P. Shor, The quantum channel capacity and coherent information, Lecture notes, *MSRI Workshop on Quantum Computation* (2002).
[5] I. Devetak, The private classical capacity and quantum capacity of a quantum channel, IEEE Trans. Inf. Theory **51**, 44 (2005).
[6] N. Cai, A. Winter, and R. W. Yeung, Quantum privacy and quantum wiretap channels, Probl. Inf. Transm. **40**, 318 (2004).
[7] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Entanglement-assisted classical capacity of noisy quantum channels, Phys. Rev. Lett. **83**, 3081 (1999).
[8] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Entanglement-assisted capacity of a quantum channel and the reverse Shannon theorem, IEEE Trans. Inf. Theory **48**, 2637 (2002).
[9] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A **53**, 2046 (1996).
[10] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of noisy entanglement and faithful teleportation via noisy channels, Phys. Rev. Lett. **76**, 722 (1996).
[11] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Mixed-state entanglement and quantum error correction, Phys. Rev. A **54**, 3824 (1996).
[12] I. Devetak and A. Winter, Distillation of secret key and entanglement from quantum states, Proc. R. Soc. A **461**, 207 (2005).
[13] L. Lami and B. Regula, Distillable entanglement under dually non-entangling operations, Nat. Commun. **15**, 10120 (2024).
[14] P. M. Hayden, M. Horodecki, and B. M. Terhal, The asymptotic entanglement cost of preparing a quantum state, J. Phys. A **34**, 6891 (2001).
[15] D. Yang, M. Horodecki, R. Horodecki, and B. Synak-Radtke, Irreversibility for all bound entangled states, Phys. Rev. Lett. **95**, 190501 (2005).
[16] F. Buscemi and N. Datta, Entanglement cost in practical scenarios, Phys. Rev. Lett. **106**, 130503 (2011).
[17] H. Yamasaki, K. Kuroiwa, P. Hayden, and L. Lami, Entanglement cost for infinite-dimensional physical systems, arXiv:2401.09554.
[18] K. G. H. Vollbrecht and R. F. Werner, Entanglement measures under symmetry, Phys. Rev. A **64**, 062307 (2001).
[19] P. W. Shor, Equivalence of additivity questions in quantum information theory, Commun. Math. Phys. **246**, 473 (2004).
[20] P. Hayden and A. Winter, Counterexamples to the maximal p-norm multiplicativity conjecture for all $p > 1$, Commun. Math. Phys. **284**, 263 (2008).
[21] T. Cubitt, A. W. Harrow, D. Leung, A. Montanaro, and A. Winter, Counterexamples to additivity of minimum output p-Rényi entropy for p close to 0, Commun. Math. Phys. **284**, 281 (2008).
[22] M. B. Hastings, Superadditivity of communication capacity using entangled inputs, Nat. Phys. **5**, 255 (2009).
[23] G. Smith and J. Yard, Quantum communication with zero-capacity channels, Science **321**, 1812 (2008).
[24] W. K. Wootters, Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett. **80**, 2245 (1998).
[25] T. Cubitt, D. Elkouss, W. Matthews, M. Ozols, D. Pérez-García, and S. Strelchuk, Unbounded number of channel uses may be required to detect quantum capacity, Nat. Commun. **6**, 6739 (2015).
[26] K. Audenaert, M. B. Plenio, and J. Eisert, Entanglement cost under positive-partial-transpose-preserving operations, Phys. Rev. Lett. **90**, 027901 (2003).

[27] X. Wang and M. M. Wilde, Cost of quantum entanglement simplified, Phys. Rev. Lett. **125**, 040502 (2020).

[28] X. Wang and M. M. Wilde, Exact entanglement cost of quantum states and channels under positive-partial-transpose-preserving operations, Phys. Rev. A **107**, 012429 (2023).

[29] G. Gour and C. M. Scandolo, Dynamical entanglement, Phys. Rev. Lett. **125**, 180505 (2020).

[30] G. Gour and C. M. Scandolo, Entanglement of a bipartite channel, Phys. Rev. A **103**, 062422 (2021).

[31] A. Peres, Separability criterion for density matrices, Phys. Rev. Lett. **77**, 1413 (1996).

[32] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, Phys. Lett. A **223**, 1 (1996).

[33] P. Horodecki, Separability criterion and inseparable mixed states with positive partial transposition, Phys. Lett. A **232**, 333 (1997).

[34] E. M. Rains, Bound on distillable entanglement, Phys. Rev. A **60**, 179 (1999).

[35] E. M. Rains, A semidefinite program for distillable entanglement, IEEE Trans. Inf. Theory **47**, 2921 (2001).

[36] S. Ishizaka, Binegativity and geometry of entangled states in two qubits, Phys. Rev. A **69**, 020301(R) (2004).

[37] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[38] M. B. Plenio, Logarithmic negativity: A full entanglement monotone that is not convex, Phys. Rev. Lett. **95**, 090503 (2005).

[39] P. Calabrese, J. Cardy, and E. Tonni, Entanglement negativity in quantum field theory, Phys. Rev. Lett. **109**, 130502 (2012).

[40] P. Calabrese, J. Cardy, and E. Tonni, Entanglement negativity in extended systems: A field theoretical approach, J. Stat. Mech. (2013) P02008.

[41] P. Calabrese, J. Cardy, and E. Tonni, Finite temperature entanglement negativity in conformal field theory, J. Phys. A **48**, 015006 (2014).

[42] Y. A. Lee and G. Vidal, Entanglement negativity and topological order, Phys. Rev. A **88**, 042318 (2013).

[43] M. Hoogeveen and B. Doyon, Entanglement negativity and entropy in non-equilibrium conformal field theory, Nucl. Phys. **B898**, 78 (2015).

[44] V. Eisler and Z. Zimboras, Entanglement negativity in the harmonic chain out of equilibrium, New J. Phys. **16**, 123020 (2014).

[45] P. Ruggiero, V. Alba, and P. Calabrese, Entanglement negativity in random spin chains, Phys. Rev. B **94**, 035152 (2016).

[46] X. Dong, J. Kudler-Flam, and P. Rath, Entanglement negativity and replica symmetry breaking in general holographic states, arXiv:2409.13009.

[47] X. Wang and M. M. Wilde, Errata for "Cost of quantum entanglement simplified" and "Exact entanglement cost of quantum states and channels under PPT-preserving operations", 10.5281/zenodo.11061607 (2024).

[48] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Entangling operations and their implementation using a small amount of entanglement, Phys. Rev. Lett. **86**, 544 (2001).

[49] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-state entanglement and distillation: Is there a "bound" entanglement in nature?, Phys. Rev. Lett. **80**, 5239 (1998).

[50] T. Eggeling, Karl Gerd H. Vollbrecht, R. F. Werner, and M. M. Wolf, Distillability via protocols respecting the positivity of partial transpose, Phys. Rev. Lett. **87**, 257902 (2001).

[51] X. Wang and R. Duan, Irreversibility of asymptotic entanglement manipulation under quantum operations completely preserving positivity of partial transpose, Phys. Rev. Lett. **119**, 180506 (2017).

[52] X. Wang, W. Xie, and R. Duan, Semidefinite programming strong converse bounds for classical capacity, IEEE Trans. Inf. Theory **64**, 640 (2018).

[53] K. Fang and H. Fawzi, Geometric Rényi divergence and its applications in quantum channel capacities, Commun. Math. Phys. **384**, 1615 (2021).

[54] M. Hayashi, *Quantum Information Theory: Mathematical Foundation*, 2nd ed., Graduate Texts in Physics (Springer, Berlin Heidelberg, 2017).

[55] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.134.090202, which includes Refs. [56–66], for details.

[56] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).

[57] D. Bruß and A. Peres, Construction of quantum states with bound entanglement, Phys. Rev. A **61**, 030301 (2000).

[58] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Math. Z. **17**, 228 (1923).

[59] M. A. Nielsen, Conditions for a class of entanglement transformations, Phys. Rev. Lett. **83**, 436 (1999).

[60] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. **70**, 1895 (1993).

[61] S. Khatri and M. M. Wilde, Principles of quantum communication theory: A modern approach, arXiv:2011.04672.

[62] M. Horodecki, Entanglement measures, Quantum Inf. Comput. **1**, 3 (2001).

[63] M. J. Donald, M. Horodecki, and O. Rudolph, The uniqueness theorem for entanglement measures, J. Math. Phys. (N.Y.) **43**, 4252 (2002).

[64] L. Lami and B. Regula, No second law of entanglement manipulation after all, Nat. Phys. **19**, 184 (2023).

[65] G. Vidal and R. Tarrach, Robustness of entanglement, Phys. Rev. A **59**, 141 (1999).

[66] C. Shannon, The zero error capacity of a noisy channel, IRE Trans. Inf. Theory **2**, 8 (1956).

[67] K. Audenaert, B. De Moor, K. G. H. Vollbrecht, and R. F. Werner, Asymptotic relative entropy of entanglement for orthogonally invariant states, Phys. Rev. A **66**, 032310 (2002).

[68] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, England, 2018).

[69] P. Skrzypczyk and D. Cavalcanti, *Semidefinite Programming in Quantum Information Science* (IOP Publishing, Bristol, UK, 2023).

[70] L. Vandenberghe and S. Boyd, Semidefinite programming, SIAM Rev. **38**, 49 (1996).

[71] H. Fawzi and O. Fawzi, Defining quantum divergences via convex optimization, Quantum **5**, 387 (2021).

[72] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Complete family of separability criteria, Phys. Rev. A **69**, 022308 (2004).

[73] M. Navascués, S. Pironio, and A. Acín, A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations, New J. Phys. **10**, 073013 (2008).

[74] A. W. Harrow, A. Natarajan, and X. Wu, An improved semidefinite programming hierarchy for testing entanglement, Commun. Math. Phys. **352**, 881 (2017).

[75] O. Fawzi, A. Shayeghi, and H. Ta, A hierarchy of efficient bounds on quantum capacities exploiting symmetry, IEEE Trans. Inf. Theory **68**, 7346 (2022).

[76] M. Berta, F. Borderi, O. Fawzi, and V. B. Scholz, Semidefinite programming hierarchies for constrained bilinear optimization, Math. Program. **194**, 781 (2022).

[77] Y. T. Lee, A. Sidford, and S. C.-W. Wong, A faster cutting plane method and its implications for combinatorial and convex optimization, in *Proceedings of the 2015 IEEE 56th Annual Symposium on Foundations of Computer Science* (2015), pp. 1049–1065.

[78] J. van Apeldoorn, A. Gilyén, S. Gribling, and R. de Wolf, Quantum SDP-Solvers: Better upper and lower bounds, Quantum **4**, 230 (2020).

End Matter

Appendix A: The task—We start by defining $E^{\mathrm{exact}}_{c,\mathrm{PPT}}$ in rigorous terms. A (quantum) channel $\Lambda\colon X\to Y$ is a completely positive and trace preserving map taking as input states of a quantum system X and outputting states of Y. The set of completely positive maps (respectively, quantum channels) from X to Y will be denoted as $\mathrm{CP}(X\to Y)$ (respectively, $\mathrm{CPTP}(X\to Y)$). If $X=AB$ and $Y=A'B'$ are both bipartite systems and $\Lambda\in\mathrm{CP}(AB\to A'B')$, we say that Λ is PPT if $\Gamma_{B'}\circ\Lambda\circ\Gamma_B$ is still completely positive, where Γ_B denotes the partial transpose on B, and analogously for $\Gamma_{B'}$. Another way to understand this class of channels is to realize that they completely preserve the set of PPT states, in the sense that $\Gamma_{B_1B_2'}[\mathrm{id}\otimes\Lambda(\sigma_{A_1B_1A_2B_2})]\geq 0$ for any state σ with $\Gamma_{B_1B_2}(\sigma_{A_1B_1A_2B_2})\geq 0$. We can then define the zero-error PPT entanglement cost of any bipartite state $\rho=\rho_{AB}$ as

$$E^{\mathrm{exact}}_{c,\mathrm{PPT}}(\rho) := \inf\big\{R\colon \text{for all sufficiently large}\, n\in\mathbb{N}\ \exists\Lambda_n\in\mathrm{PPT}\cap\mathrm{CPTP}\colon \Lambda_n(\Phi_2^{\otimes\lfloor Rn\rfloor})=\rho^{\otimes n}\big\}. \tag{A1}$$

Here, $\Phi_2:=|\Phi_2\rangle\langle\Phi_2|$, where $|\Phi_2\rangle:=(1/\sqrt{2})(|00\rangle+|11\rangle)$ is the two-qubit maximally entangled state, i.e., the "ebit," and Λ_n is required to be a PPT channel.

Appendix B: The quantifier E_κ—Wang and Wilde [27,28,47] introduced and studied the SDP-computable quantity

$$E_\kappa(\rho) := \log_2 \min\{\mathrm{Tr}S\colon\ -S\leq\rho^\Gamma\leq S,\ S^\Gamma\geq 0\}. \tag{B1}$$

Among other things, they showed that (i) E_κ is monotonically nonincreasing under PPT channels; (ii) $E_\kappa(\rho)\geq E_N(\rho)$ for all states ρ, with equality when ρ has zero binegativity; (iii) E_κ is subadditive, meaning that

$$E_\kappa(\rho\otimes\rho')\leq E_\kappa(\rho)+E_\kappa(\rho') \tag{B2}$$

for all pairs of states $\rho=\rho_{AB}$ and $\rho'=\rho'_{A'B'}$; and (iv) its regularization yields the zero-error PPT entanglement cost, i.e.,

$$E^{\mathrm{exact}}_{c,\mathrm{PPT}}(\rho)=E_\kappa^\infty(\rho) := \lim_{n\to\infty}\frac{1}{n}E_\kappa(\rho^{\otimes n}). \tag{B3}$$

As said, it was claimed in [27] that E_κ is additive, meaning that equality holds in (B2). However, this claim is incorrect (see also [47]). To disprove it, it is useful to first note that additivity indeed holds when both ρ and ρ' have zero binegativity, simply because in that case E_κ coincides with the logarithmic negativity by property (ii), and this latter measure *is* additive, as one sees immediately by looking at its definition in terms of the 1-norm of the partially transposed state. Hence, our search for a counterexample must start with the construction of states with nonzero binegativity. That such states do exist was reported already in [36,67] based on numerical evidence. However, here we present a simpler, analytical construction.

Appendix C: Punch card states—Let $A\geq 0$ be a positive semidefinite $d\times d$ matrix, and let Q be another $d\times d$ symmetric matrix with only 0/1 entries and all 1's on the main diagonal (i.e., such that $Q_{ii}=1$ for all i). The associated "punch card state" is the bipartite quantum state on $\mathbb{C}^d\otimes\mathbb{C}^d$ defined by

$$\pi_{A,Q} := \frac{1}{N_{A,Q}}\left(\sum_{i,j}A_{ij}|ii\rangle\langle jj| + \sum_{i\neq j}Q_{ij}|A_{ij}||ij\rangle\langle ij|\right), \tag{C1}$$

where $N_{A,Q}$ is chosen so that $\mathrm{Tr}\pi_{A,Q}=1$. It can be verified that

$$\pi^\Gamma_{A,Q}\simeq\frac{1}{N_{A,Q}}\left[\left(\sum_i A_{ii}|ii\rangle\langle ii|\right)\oplus\bigoplus_{i<j}\begin{pmatrix} Q_{ij}|A_{ij}| & A_{ij}\\ A^*_{ij} & Q_{ij}|A_{ij}|\end{pmatrix}\right], \tag{C2}$$

from which, using the identity

$$\left|\begin{pmatrix} Q_{ij}|A_{ij}| & A_{ij}\\ A^*_{ij} & Q_{ij}|A_{ij}|\end{pmatrix}\right| = \begin{pmatrix} |A_{ij}| & Q_{ij}A_{ij}\\ Q_{ij}A^*_{ij} & |A_{ij}|\end{pmatrix}, \tag{C3}$$

valid because $Q_{ij} \in \{0,1\}$, one derives that

$$|\pi_{A,Q}^{\Gamma}|^{\Gamma} = \frac{1}{N_{A,Q}}\left(\sum_{i,j} Q_{ij}A_{ij}|ii\rangle\langle jj| + \sum_{i\neq j}|A_{ij}||ij\rangle\langle ij|\right). \tag{C4}$$

Therefore, if A and Q are chosen such that $Q\circ A \not\geq 0$, where $\circ$ denotes the Hadamard (i.e., entry-wise) product between matrices, $\pi_{A,Q}$ will have nonzero binegativity, i.e., $|\pi_{A,Q}^{\Gamma}|^{\Gamma} \not\geq 0$. It is easy to construct examples of A and Q that meet the above criteria, the simplest one being

$$A_0 \coloneqq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad Q_0 \coloneqq \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{C5}$$

Having constructed a state with nonzero binegativity, we can wonder whether two copies of it already violate the additivity of E_κ. And sure enough, they do.

Lemma 1—The punch card state $\pi_0 \coloneqq \pi_{A_0,Q_0}$ defined by (C1) with the substitution (C5) satisfies

$$1.001 \approx E_\kappa(\pi_0^{\otimes 2}) < 2E_\kappa(\pi_0) \approx 1.029. \tag{C6}$$

In particular, the subadditivity of E_κ and Lemma 1 imply that $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\pi_0) \leq \frac{1}{2}E_\kappa(\pi_0^{\otimes 2}) < E_\kappa(\pi_0)$ and therefore that $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\pi_0) \neq E_\kappa(\pi_0)$, thus invalidating the main claim of the works [27,28].

Appendix D: Two SDP hierarchies—Recall that in (5) and (6) we introduced the χ hierarchy $E_{\chi,p}(\rho) = \log_2 \chi_p(\rho)$ with

$$\chi_p(\rho) \coloneqq \min\left\{\mathrm{Tr}S_p :\ -S_i \leq S_{i-1}^{\Gamma} \leq S_i,\ i=0,\ldots,p,\ S_{-1}=\rho\right\}, \tag{D1}$$

which constitutes a generalization of the logarithmic negativity. We now introduce another complementary hierarchy of SDPs, the "κ hierarchy," defined for $q \in \mathbb{N}^+$ $(q \geq 1)$ by

$$E_{\kappa,q}(\rho) \coloneqq \log_2 \kappa_q(\rho), \tag{D2}$$

with

$$\kappa_q(\rho) \coloneqq \min\{\mathrm{Tr}S_{q-1}:\ -S_i \leq S_{i-1}^{\Gamma} \leq S_i,\ i=0,\ldots,q-1,\ S_{-1}=\rho,\ S_{q-1}^{\Gamma} \geq 0\}. \tag{D3}$$

Observe the resemblance to the definition of χ_p in (D1): the only difference between the two optimizations is the condition $S_{p-1}^{\Gamma} \geq 0$; adding that to (D1) yields immediately (D3) with $q \mapsto p$, and indeed in that case the optimal S_p would automatically be $S_p = S_{p-1}^{\Gamma}$. Furthermore, $E_{\kappa,1} = E_\kappa$ coincides with the quantity (B1) introduced by Wang and Wilde, of which the κ hierarchy thus constitutes a generalization.

In the SM [55] we explore the properties of the quantities $E_{\chi,p}$ and $E_{\kappa,q}$, showing them to be legitimate entanglement measures. In particular, the functions are all suitably normalized, continuous, faithful on PPT states, and strongly monotonic under PPT operations. The pivotal property that distinguishes $E_{\chi,p}$ from $E_{\kappa,q}$ is that, while the quantities $E_{\kappa,q}$ are only subadditive, the χ quantities are fully additive under tensor products, meaning that regularization can always be avoided.

Two key insights lead to the proof of our main results. First, that the two hierarchies provide complementary bounds on $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$, the fundamental quantity we want to estimate: namely, the χ hierarchy gives increasing lower bounds on $E_{c,\mathrm{PPT}}^{\mathrm{exact}}$, while the κ hierarchy gives decreasing upper bounds on it. In other words, on any fixed state ρ

$$E_{\chi,0} \leq E_{\chi,1} \leq \ldots \leq E_{c,\mathrm{PPT}}^{\mathrm{exact}} = E_\kappa^\infty \leq \ldots \leq E_{\kappa,2} \leq E_{\kappa,1}. \tag{D4}$$

In particular, $E_{\chi,p}$ is increasing in p, while $E_{\kappa,q}$ is decreasing in q. Equation (D4) immediately shows the remarkable connection between two very different limits: one in the number of copies n, which is needed to compute E_κ^∞ [see Eq. (B3)], and one in the hierarchy levels p and q.

The second insight is that there is a connection between the χ and κ hierarchies, as expressed by the following key technical result, proven in the SM [55].

Proposition 1—For all states $\rho = \rho_{AB}$ on a system of minimal local dimension $d \coloneqq \min\{|A|,|B|\} \geq 2$, and all $p \in \mathbb{N}^+$,

$$\kappa_p(\rho) \leq \frac{d}{2}\chi_p(\rho) - \left(\frac{d}{2}-1\right)\chi_{p-1}(\rho). \tag{D5}$$

With Proposition 1 at hand, we can now see how it implies our two main results, Theorems 1 (exact expression for the cost) and 2 (time complexity of computing the cost).

Proof sketch of Theorem 1—Combining (D4) and (D5) shows that

$$2^{E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)} \leq \kappa_p(\rho) \leq \frac{d}{2}\chi_p(\rho) - \left(\frac{d}{2}-1\right)\chi_{p-1}(\rho). \tag{D6}$$

The quantity that we are really interested in, however, is the normalized difference between $\chi_p(\rho)$ and its claimed limiting value $2^{E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)}$. To see what the above inequality tells us in this respect, we can define the quantity

$$\varepsilon_p(\rho) \coloneqq 1 - \frac{\chi_p(\rho)}{2^{E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)}}, \tag{D7}$$

by means of which (D6) can be cast as $\varepsilon_p(\rho) \leq \left(1 - \frac{2}{d}\right)\varepsilon_{p-1}(\rho)$. Iterating the above relation gives

$$\varepsilon_p(\rho) \leq \left(1 - \frac{2}{d}\right)^p \varepsilon_0(\rho) \leq \left(1 - \frac{2}{d}\right)^p, \tag{D8}$$

which entails (7) after elementary algebraic manipulations. Taking the limit $p \to \infty$ in (7) proves also (8). ■

Proof sketch of Theorem 2—It suffices to formalize the qualitative argument provided below the statement of the theorem. For $d > 2$, we first choose

$$p_d \coloneqq \left\lceil \frac{\log_2(2d/\varepsilon)}{-\log_2(1 - \frac{2}{d})} \right\rceil = \mathcal{O}(d \log(d/\varepsilon)), \tag{D9}$$

so that, with the notation in (D7), $\varepsilon_{p_d}(\rho) \leq (\varepsilon/2d)$. Using $\ln(2)|a - b| \leq |2^a - 2^b|$, valid for a, $b \geq 0$, we get

$$0 \leq E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) - E_{\chi,p_d}(\rho) \leq \frac{2^{E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho)}\varepsilon}{2d \ln 2} \leq \frac{\varepsilon}{2 \ln 2}, \tag{D10}$$

where the last inequality is a consequence of the fact that every state can be created via a quantum teleportation protocol—and hence with PPT operations—from a maximally entangled state, which entails that $E_{c,\mathrm{PPT}}^{\mathrm{exact}}(\rho) \leq \log_2 d$ for all ρ. We then solve the SDP for $\chi_{p_d}(\rho)$ up to an additive error $(\ln 2 - 1/2)\varepsilon$ by running an optimized SDP solver [77,78]. Doing so yields an approximation of $E_{\chi,p_d}(\rho)$ up to an additive error $[1 - 1/(2 \ln 2)]\varepsilon$. Adding this up with the error in (D10) yields a total error of ε. The time complexity in (10) can be calculated using known theoretical bounds on the complexity of SDPs, e.g., those found in [77]. ■