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Distillable entanglement under dually non-entangling operations

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Computing the exact rate at which entanglement can be distilled from noisy quantum states is one of the longest-standing questions in quantum information. We give an exact solution for entanglement distillation under the set of dually non-entangling (DNE) operations—a relaxation of the typically considered local operations and classical communication, comprising all channels which preserve the sets of separable states and measurements. We show that the DNE distillable entanglement coincides with a modified version of the regularised relative entropy of entanglement in which the arguments are measured with a separable measurement. Ours is only the second known regularised formula for the distillable entanglement under any class of free operations in entanglement theory, after that given by Devetak and Winter for (one-way) local operations and classical communication. An immediate consequence of our finding is that, under DNE, entanglement can be distilled from any entangled state. As our second main result, we construct a general upper bound on the DNE distillable entanglement, using which we prove that the separably measured relative entropy of entanglement can be strictly smaller than the regularisation of the standard relative entropy of entanglement, solving an open problem posed by Li and Winter. Finally, we study also the reverse task of entanglement dilution and show that the restriction to DNE operations does not change the entanglement cost when compared with the larger class of non-entangling operations. This implies a strong form of irreversiblility of entanglement theory under DNE operations: even when asymptotically vanishing amounts of entanglement may be generated, entangled states cannot be converted reversibly.

Entanglement distillation is poised to be one of the fundamental primitives of the emerging field of quantum technologies[1–3]. Its goal is to transform many copies of a noisy entangled state ρ_{AB} emitted by some source into (a smaller number of) *ebits*, i.e., perfect units of entanglement, that can then be used as fuel in a variety of quantum protocols, e.g., teleportation[4], dense coding[5], or quantum key distribution[6,7], as well as to demonstrate violations of Bell inequalities[8]. To perform this transformation, one is allowed to use quantum operations taken from a class of *free operations* $\mathcal{F}$, assumed to comprise all quantum channels that are easy to implement on our quantum devices with the resources at our disposal. The fundamental figure of merit that characterises the ultimate efficiency of entanglement distillation on ρ_{AB} is the *distillable entanglement* under $\mathcal{F}$, denoted by $E_{d,\mathcal{F}}(\rho_{AB})$. This depends critically both on the input

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state ρ_{AB} and on the set of free operations $\mathcal{F}$ with which distillation is carried out.

Historically, the first class of free operations to be widely studied was that of *local operations and classical communication* (LOCC). LOCCs are the most general operations that can be performed by spatially separated parties that are bound to the rules of quantum mechanics and can communicate only classically[9]. In spite of their operational importance, LOCCs are exceedingly difficult to characterise mathematically, and as a consequence we currently do not possess any formula to compute the LOCC distillable entanglement on arbitrary states, or even decide when it is zero and when it is not[10,11]. This is to be contrasted with the fact that we *do* possess a formula for the LOCC entanglement *cost*, which is given by the regularised entanglement of formation[12].

To bypass this problem, in recent years there has been an increasing interest in characterising entanglement manipulation under classes of free operations that approximate LOCCs. There are at least two valid reasons to pursue this goal, besides the obvious one that simpler classes of free operations offer an ideal test-bed to improve our understanding of entanglement manipulation. First, in this way one can obtain lower and upper bounds on the LOCC distillable entanglement, because if $\mathcal{F}_1 \subseteq \mathrm{LOCC} \subseteq \mathcal{F}_2$ then $E_{d,\mathcal{F}_1}(\rho_{AB}) \le E_{d,\mathrm{LOCC}}(\rho_{AB}) \le E_{d,\mathcal{F}_2}(\rho_{AB})$. Several of the best known bounds on the distillable entanglement can be seen as stemming from this approach. For example, the restriction to one-way LOCCs yields the hashing lower bound [ref. 13, Theorem 13], while going to the larger set of PPT operations results in the upper bounds via the negativity[14,15] and the Rains bound[16]. Nevertheless, even under simpler classes of operations, the ultimate capabilities of entanglement manipulation—and in particular the exact value of distillable entanglement—are not known, with the sole exception of one-way LOCC[13].

But there is also another, more fundamental reason to go beyond the LOCC paradigm. Since the early days of entanglement theory, a whole line of research[17–19] has focused on the conceptually striking parallels between the laws of entanglement manipulation, which govern the interconversion of pure and mixed entanglement, and those of thermodynamics, which govern the interconversion of work and heat. In building a 'thermodynamic theory of entanglement', a top-down, axiomatic approach is somewhat preferable to the bottom-up approach that leads to the definition of LOCC, and arguably closer in spirit to the original founding principles of thermodynamics[20–22]. This approach has been already quite fruitful, shedding light on the fundamental question of (ir)reversibility of entanglement manipulation[23–26], but many of the issues it raises remain unresolved.

In this work, we focus on a particular superset of LOCC, namely the *dually non-entangling* (DNE) operations. DNE operations, originally introduced by Chitambar et al.[27], can be easily understood from a rather natural axiomatic perspective as those that preserve separability (i.e., absence of entanglement) of states when seen in the Schrödinger picture *and* of measurements when seen in the Heisenberg picture. An important point is that DNE protocols are more restricted than some other commonly employed operations such as non-entangling (NE) maps[23,24], which means that they provide a closer approximation to LOCCs and can potentially yield improved bounds on their operational processing power. Here we completely characterise asymptotic entanglement manipulation under DNE operations, connecting it with the important operational task of entanglement testing, and revealing new features of phenomena such as entanglement irreversibility and bound entanglement.

Results

Our contribution

Our first main result is a clean formula for the DNE distillable entanglement (Theorem 1), which can be expressed as a regularised separably measured divergence $D^{\mathbb{SEP},\infty}(\cdot\|\mathcal{S})$ — a modified version of the relative entropy of entanglement $D^{\infty}(\cdot\|\mathcal{S})$[28,29] that was originally introduced by Piani in a seminal work[30] and has recently found some applications[26,31–33]. We dub it the Piani relative entropy of entanglement. To compute it, one calculates the distance of a given state from the set of separable states as measured by a form of measured relative entropy that involves an optimisation over separable measurements only. In spite of the seemingly more complicated definition, the resulting quantity enjoys a plethora of desirable properties, some of which, such as strong super-additivity [ref. 30, Theorem 2], are often very useful[30,34,35] but do not hold for the standard relative entropy of entanglement[36]. As a by-product of our result, many of these properties are inherited by the DNE distillable entanglement. Perhaps the most important one is the *faithfulness* — since the Piani relative entropy is non-zero for all entangled states, this immediately implies that the phenomenon of bound entanglement does not exist under DNE operations.

Our result represents one of the few instances where the distillable entanglement under some relevant class of free operations can be calculated, albeit in terms of some regularised expression. *Regularisation* here means that one needs to understand the limiting behaviour of the given quantity when acting on many copies of a quantum state. The necessity for such a procedure is a consequence of the fact that operational quantities encountered in quantum information are typically not additive[36–41], precluding the validity of non-regularised ('single-letter') expressions. We should remark here that while regularised formulas do not allow for a straightforward calculation because of the ubiquity of non-additivity phenomena in quantum information, they are nevertheless amenable to theoretical investigation, which is the reason why many cornerstone results in quantum information theory, such as the Lloyd–Shor–Devetak theorem[42–44], or the Holevo–Schumacher–Westmoreland theorem[45,46], are proofs of regularised formulas. Besides our result, the only other known formula for distillable entanglement of *any* kind is the result of Devetak and Winter[13], where a regularised expression is found for the rate of distillation under local operations assisted by one- or two-way classical communication; however, the expression there involves complex optimisation problems, meaning that its computability and applicability is questionable. A regularised expression has also been conjectured for the distillable entanglement under the class of non-entangling operations[24,26] in terms of $D^{\infty}(\cdot\|\mathcal{S})$ — this conjecture being equivalent to the notorious generalised quantum Stein's lemma[47,48].

To complement our study of entanglement distillation, we also provide a comprehensive characterisation of the reverse task of entanglement dilution under DNE operations. The relevant quantity here is the entanglement cost $E_{c,\mathcal{F}}(\rho_{AB})$, that is, the rate at which maximally entangled states are needed in order to produce a given noisy state. We show that the entanglement cost under DNE operations is the same as the one under the larger class of non-entangling operations. On the one hand, this means that it suffices to employ the smaller and more restrictive set of DNE operations to achieve optimal dilution rates. On the other hand, the result allows us to use a number of previously established findings in the study of NE operations[24,25] to directly constrain or even exactly compute the DNE entanglement cost. We will see this to have crucial consequences.

One of the most important application of the axiomatic approaches to entanglement theory has been the study of entanglement reversibility[23–26], namely the question of whether the rate of entanglement distillation $E_{d,\mathcal{F}}(\rho_{AB})$ equals the entanglement cost $E_{c,\mathcal{F}}(\rho_{AB})$. The fundamental consequence of such reversibility would the establishment of a 'second law' of entanglement: asymptotic entanglement transformations would be completely governed by a single function, playing a role analogous to entropy in thermodynamics. So far, reversibility has not been shown under any class of quantum channels[26], but there is a strong contender that has been conjectured

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to yield a reversible entanglement theory: namely, asymptotically non-entangling (ANE) maps[23,24]. Although entanglement is known to be irreversible under non-entangling channels[25], the entanglement manipulation rates can be increased by allowing small amounts of entanglement (according to some appropriate entanglement measure) to be generated in the process, as long as they vanish asymptotically. Using our characterisation of entanglement dilution under DNE maps, we can show that the entanglement cost under DNE maps enhanced by such asymptotic entanglement generation (which we may refer to as asymptotically dually non-entangling maps, ADNE) equals the corresponding entanglement cost under ANE maps (Theorem 2). But the latter is already known[24]: it is given exactly by the regularised relative entropy of entanglement $D^{\infty}(\cdot\|\mathcal{S})$.

We thus obtain a complete understanding of the theory of entanglement manipulation under asymptotically dually non-entangling channels: the distillable entanglement is given by the regularised Piani relative entropy $D^{\mathbb{SEP},\infty}(\cdot\|\mathcal{S})$, while the entanglement cost by the standard relative entropy of entanglement $D^{\infty}(\cdot\|\mathcal{S})$. The question of reversibility of entanglement under ADNE operations is thus reduced to the question of whether there exists a gap between the quantities $D^{\infty}(\cdot\|\mathcal{S})$ and $D^{\mathbb{SEP},\infty}(\cdot\|\mathcal{S})$. A very closely related question was previously asked by Li and Winter [ref. 32, Figure 1]. With our third main result (Theorem 3) we solve both of these problems, proving that the antisymmetric state α_d satisfies $D^{\mathbb{SEP},\infty}(\alpha_d\,\|\,\mathcal{S}) < D^{\infty}(\alpha_d\,\|\,\mathcal{S})$ for large enough d, namely $d \geq 13$. Our results thus directly prove that entanglement manipulation is irreversible under DNE operations, even if asymptotic entanglement generation is allowed. Such a strong form of irreversibility contrasts with the conjectured reversible framework under ANE operations[24,26] and shows that the restriction to DNE operations carries strong physical consequences. Our result also solves the open problem by Li and Winter[32] and further reinforces the fame of the antisymmetric state as the 'universal counterexample' in quantum entanglement theory[49].

Notation

A quantum state ρ_{AB} on a finite-dimensional bipartite quantum system AB is called *separable* if it can be written as $\rho_{AB} = \sum_x p_x\, \alpha_x^A \otimes \beta_x^B$, and it is called *entangled* otherwise. We denote the set of separable states on AB by $\mathcal{S}(A:B)$, or simply $\mathcal{S}$ if there is no ambiguity concerning the underlying system. We will also be interested in the set of *separable measurements*, denoted by $\mathbb{SEP}(A:B)$ or simply $\mathbb{SEP}$. It comprises all positive operator-valued measures (POVM) $(E_x)_x$ such that $E_x \in \mathrm{cone}(\mathcal{S})$ for all x, where $\mathrm{cone}(\mathcal{S})$ denotes the cone of un-normalised separable operators. Notably, all LOCC measurements are separable, but the converse is not true[9].

We can quantify the entanglement of a state ρ_{AB} by calculating its distance from the set of separable states as measured by some quantum divergence. The two main choices of quantum divergence we will be concerned with here are the *quantum (Umegaki) relative entropy*[50], given by $D(\rho\,\|\,\sigma) := \mathrm{Tr}\left[\rho(\log_2\rho - \log_2\sigma)\right]$, and the *Piani relative entropy*, which is defined for two arbitrary bipartite states $\rho = \rho_{AB}$ and $\sigma = \sigma_{AB}$ by[30]

$$D^{\mathbb{SEP}}(\rho\|\sigma) := \sup_{(E_x)_x \in \mathbb{SEP}} \sum_x \mathrm{Tr}[\rho E_x] \log_2 \frac{\mathrm{Tr}[E_x\rho]}{\mathrm{Tr}[E_x\sigma]}. \tag{1}$$

Yet another possible choice is the *max-relative entropy*, defined by $D_{\max}(\rho\,\|\,\sigma) := \min\{\lambda :\ \rho \leq 2^{\lambda}\sigma\}$. The *relative entropy of entanglement*, the *Piani relative entropy of entanglement*, and the *max-relative entropy of entanglement* are then defined by[28–30]

$$\mathbb{D}(\rho\|\mathcal{S}) := \min_{\sigma\in\mathcal{S}} \mathbb{D}(\rho\|\sigma), \qquad \mathbb{D} = D,\, D^{\mathbb{SEP}},\, D_{\max}. \tag{2}$$

In quantum information one often looks at the asymptotic limit of many copies, which captures the ultimate limitations to which quantum phenomena are subjected, and is reminiscent of the thermodynamic limit in statistical physics. Doing so yields the *regularised* entanglement measures

$$\mathbb{D}^{\infty}(\rho\|\mathcal{S}) := \lim_{n\to\infty} \frac{1}{n}\, \mathbb{D}\left(\rho^{\otimes n}\ \|\ \mathcal{S}_n\right), \quad \mathbb{D} = D,\, D^{\mathbb{SEP}},\, D_{\max}. \tag{3}$$

where $\mathcal{S}_n = \mathcal{S}(A^n : B^n)$. Because of the hierarchical relations existing between the corresponding divergences, it holds that

$$D^{\mathbb{SEP}}(\rho\|\mathcal{S}) \leq D(\rho\|\mathcal{S}) \leq D_{\max}(\rho\|\mathcal{S}), \tag{4}$$

and analogously for the regularised quantities. These three entanglement measures have however remarkably different features. While $D(\rho\,\|\,\mathcal{S})$ and $D_{\max}(\rho\,\|\,\mathcal{S})$ are sub-additive on several independent systems, $D^{\mathbb{SEP}}(\rho\,\|\,\mathcal{S})$ is not only super-additive but actually *strongly super-additive*, meaning that [ref. 30, Theorem 2]

$$\begin{aligned} &D^{\mathbb{SEP}}\left(\rho_{AA'BB'}\ \|\ \mathcal{S}(AA':BB')\right) \\ &\geq D^{\mathbb{SEP}}\left(\rho_{AB}\ \|\ \mathcal{S}(A:B)\right) + D^{\mathbb{SEP}}\left(\rho_{A'B'}\ \|\ \mathcal{S}(A':B')\right) \end{aligned} \tag{5}$$

for all (possibly correlated) four-partite states $\rho_{AA':BB'}$.

For a given class of *free operations* $\mathcal{F}$ and some bipartite state ρ_{AB}, the corresponding *distillable entanglement* $E_{d,\mathcal{F}}(\rho_{AB})$ can be defined as the maximum number of ebits that can be extracted per copy of ρ_{AB}, in the asymptotic limit where many copies are available, using operations from $\mathcal{F}$ while making a vanishingly small error. Although $\mathcal{F} = \mathrm{LOCC}$ is the traditional choice[2], for the aforementioned reasons we are interested here in the larger class of dually non-entangling operations, previously introduced by Chitambar et al.[27] This can be justified axiomatically as follows. Any free operation $\Lambda = \Lambda_{AB\to A'B'}$ that defines a valid entanglement manipulation protocol should transform separable states into separable states, i.e., it should not inject additional entanglement into the system. Operations that satisfy this constraint are called *non-entangling* (NE)[23,25]. While this appropriately captures what should happen in the Schrödinger picture where Λ acts on states, we can also look at the Heisenberg picture, where $\Lambda^{\dagger}$, the adjoint defined by $\mathrm{Tr}\, X\Lambda(Y) = \mathrm{Tr}\Lambda^{\dagger}(X)Y$, acts on measurement operators. If we also assume that pre-processing by Λ should not turn a separable measurement into a non-separable one, we obtain the set of *dually non-entangling* (DNE) operations, defined by the conditions[27]

$$\begin{aligned} \Lambda(\sigma_{AB}) &\in \mathcal{S}(A':B') & \forall\, \sigma_{AB} &\in \mathcal{S}(A:B), \\ \Lambda^{\dagger}(E_{A'B'}) &\in \mathrm{cone}(\mathcal{S}(A:B)) & \forall\, E_{A'B'} &\in \mathrm{cone}\left(\mathcal{S}(A':B')\right). \end{aligned} \tag{6}$$

Generalising the concept of asymptotically non-entangling operations (ANE) found in the framework of Brandão and Plenio[23,24], we can also consider *asymptotically DNE* (ADNE) operations, in which we allow the creation of some entanglement, as long as this amount is sufficiently small: it must vanish asymptotically according to some fixed entanglement measure. Following Brandão and Plenio[23,24], we quantify the generated entanglement by the max-relative entropy of entanglement, replacing the first line of (6) by the condition that $\max_{\sigma\in\mathcal{S}} D_{\max}\left(\Lambda_n(\sigma)\,\|\,\mathcal{S}\right) \leq \eta_n$ for the operation Λ_n used for entanglement distillation at the n-copy level, and then requiring that $\eta_n \longrightarrow_{n\to\infty} 0$. We will denote the corresponding distillable entanglement by $E_{d,\,\mathrm{ADNE}}(\rho_{AB})$.

Distillable entanglement

We will now state our first main result, which establishes a regularised formula for the distillable entanglement under (A)DNE operations.

***Theorem* 1.** For all bipartite states $\rho = \rho_{AB}$, the distillable entanglement under (asymptotically) dually non-entangling operations coincides with the regularised Piani relative entropy of entanglement; i.e.,

$$E_{d,\,\mathrm{DNE}}(\rho) = E_{d,\,\mathrm{ADNE}}(\rho) = D^{\mathbb{SEP},\infty}(\rho \parallel \mathcal{S}). \tag{7}$$

As a consequence, there is no bound entanglement under DNE: for any entangled state ρ, it holds that

$$E_{d,\,\mathrm{DNE}}(\rho) \geq D^{\mathbb{SEP}}(\rho \parallel \mathcal{S}) > 0. \tag{8}$$

Before we sketch the main proof ideas, several remarks are in order.

(I) Eq. (8) provides a faithful single-letter lower bound on the DNE distillable entanglement of any state, and in particular, thanks to the faithfulness of the Piani relative entropy[30], it reveals that there is no bound entanglement under DNE operations. Distillability under these operations was previously studied in [ref. 27, Theorem 13], although a complete proof that all entangled states are distillable was not obtained there. Theorem 1 also provides a conceptually cleaner proof of our previous result [ref. 26, Proposition 7], where we showed that there is no bound entanglement under ANE operations; it suffices to observe that since ADNE operations are in particular ANE, we have that $E_{d,\,\mathrm{ANE}}(\rho) \geq E_{d,\,\mathrm{ADNE}}(\rho) = E_{d,\,\mathrm{DNE}}(\rho)$, and we just observed that the right-hand side is non-zero for all entangled states ρ.

(II) The result of Eq. (7) shows that the choice between DNE or ADNE operations makes no difference as far as distillation is concerned. This is a priori unexpected, as ADNE operations are strictly more powerful than DNE in general, but not entirely surprising, as the same happens for non-entangling vs. asymptotically non-entangling operations [ref. 25, Lemma S17]. Furthermore, in Supplementary Note II.B we prove that even allowing ADNE operations to generate an arbitrary sub-linear amount of entanglement (i.e., $\eta_n = o(n)$) does not increase the distillation rate.

(III) Explicitly characterising various features of distillable entanglement, such as its additivity properties, is often extremely difficult. However, equating the rate of distillation with an entropic quantity helps mitigate such problems. Thanks to Theorem 1, the (A)DNE distillable entanglement is seen to inherit all the useful properties of the Piani relative entropy of entanglement. A notable example is the strong super-additivity (5) − this feature of the (A)DNE distillable entanglement, highly non-trivial to see from its definition, establishes a strong parallel with the LOCC distillable entanglement, which is also known to be strongly super-additive [ref. 51, Table 3.4].

(IV) The presence of regularisation in the formula (7) may make the quantity look dauntingly difficult to compute in practice. However, there are several ways in which it may be efficiently estimated. First, the super-additivity of the Piani relative contrasts with the *sub*-additivity of the standard relative entropy of entanglement, which means that obtaining lower bounds for the former is as easy as upper bounds for the latter: it suffices to evaluate it on a single copy of a quantum state, yielding $D(\rho \parallel \mathcal{S}) \geq E_{d,\,\mathrm{DNE}}(\rho) \geq D^{\mathbb{SEP}}(\rho \parallel \mathcal{S})$. Furthermore, we will shortly introduce a general single-letter bound, which will often allow for an efficient evaluation of $D^{\mathbb{SEP},\infty}(\rho \parallel \mathcal{S})$ in practice.

Proof sketch of Theorem 1 and connection with hypothesis testing

Our approach will be to first establish a tight bound on the *one-shot* DNE distillable entanglement, that is, on the amount of entanglement that can be extracted from a single copy of a state ρ. By studying the asymptotic behaviour of this quantity when many copies of ρ are available, we will then obtain a formula for the rate of distillation.

To this end, we will relate DNE entanglement distillation with a seemingly different task, namely, hypothesis testing of entangled states − or *entanglement testing* for short[47,48]. In this latter task, given an unknown quantum state which could be either the entangled state ρ or any separable state, the goal is to perform a measurement and guess which of the two hypotheses is true. Our setting is different from that of refs. 47,48 in that we assume that the only allowed measurements are separable, i.e., they do not carry any entanglement themselves. In the asymmetric setting, one assumes that the probability of mistaking ρ for a separable state is at most ε and asks about the opposite error probability. The least achievable error can then be quantified by a quantity known as the separably measured hypothesis testing relative entropy[34], which is defined as

$$D_H^{\mathbb{SEP},\,\varepsilon}(\rho \parallel \sigma) := -\log_2 \inf_{(E,\,1-E)\in\mathbb{SEP}} \left\{ \mathrm{Tr}\, E\sigma : \ \mathrm{Tr}\, E\rho \geq 1-\varepsilon \right\}. \tag{9}$$

Returning to entanglement distillation, let us denote by $E^{(1),\,\varepsilon}_{d,\,\mathrm{DNE}_\eta}(\rho)$ the number of ebits that can be distilled from a single copy of a bipartite state $\rho = \rho_{AB}$, up to error ε and up to η of generated entanglement. Some thoughtful manipulation − the details of which we defer to Supplementary Note II − then shows that

$$\begin{aligned} \left\lfloor D_H^{\mathbb{SEP},\,\varepsilon}(\rho \parallel \mathcal{S}) \right\rfloor &\leq E^{(1),\,\varepsilon}_{d,\,\mathrm{DNE}_\eta}(\rho) \\ &\leq D_H^{\mathbb{SEP},\,\varepsilon}(\rho \parallel \mathcal{S}) + \eta + 1. \end{aligned} \tag{10}$$

This tells us that the DNE distillable entanglement is, up to some small terms that will asymptotically vanish, completely determined by $D_H^{\mathbb{SEP},\,\varepsilon}$. This connection between DNE entanglement distillation and entanglement testing with separable measurements is not only one of our main conceptual contributions, but on the more practical side it allows us to characterise the properties of entanglement distillation by using results from hypothesis testing. And indeed, the asymptotic behaviour of $D_H^{\mathbb{SEP},\,\varepsilon}$ was already studied in ref. 34, where it was shown that

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} D_H^{\mathbb{SEP},\,\varepsilon}\left(\rho^{\otimes n} \| \mathcal{S}_n\right) = D^{\mathbb{SEP},\infty}(\rho \| \mathcal{S}). \tag{11}$$

By Eq. (10), the expression on the left-hand side is precisely the asymptotic (A)DNE distillable entanglement, from which Eq. (7) follows. Eq. (8) is then a consequence of known properties of $D^{\mathbb{SEP}}$[30]. This concludes the sketch of the proof of Theorem 1. The complete technical details of all of our proofs can be found in the Supplementary Information.

The idea of using entanglement testing to provide bounds for distillable entanglement has a long history[29,52], but it was not until the work of Brandão and Plenio[24] that a precise equivalence between the two concepts was established in a suitable axiomatic setting. Specifically, they showed that the rate of distillation under non-entangling operations can be expressed using a related hypothesis testing problem as

$$E_{d,\,\mathrm{NE}}(\rho) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} D_H^{\mathbb{ALL},\,\varepsilon}\left(\rho^{\otimes n} \parallel \mathcal{S}_n\right), \tag{12}$$

where the notation $\mathbb{ALL}$ refers to all measurements being allowed. There is, however, a crucial difference between Brandão and Plenio's work[24] and ours: while the former constructed a one-to-one mapping between non-entangling protocols and *global* entanglement tests, the proof of our Theorem 1, and in particular that of Eq. (10), shows that DNE distillation protocols − including those that are not implementable with LOCC − are in one-to-one correspondence with *separable* entanglement tests. Our result is significant because, in practice, separable measurements are easier to implement experimentally in a distant local laboratory setting in which Alice and Bob do not have access to unbounded quantum communication.

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Entanglement cost

It is also of interest to understand the task opposite to distillation, the dilution of entanglement – that is, the use of ebits to produce a target (mixed) quantum state. The rate at which this can be done is known as the *entanglement cost* $E_{c,\mathcal{F}}(\rho_{AB})$, and the question of reversibility of entanglement manipulation[19,23–26,53] asks whether $E_{d,\mathcal{F}}(\rho_{AB}) = E_{c,\mathcal{F}}(\rho_{AB})$ holds for all states under the given class of free operations $\mathcal{F}$.

Our second main result establishes an equivalence between (A) DNE and the larger class of (A)NE operations in entanglement dilution, showing that any dilution task that could be achieved with Brandão and Plenio's (A)NE operations[24,25] can also be accomplished with the more axiomatically meaningful (A)DNE operations.

***Theorem* 2.** For all bipartite states $\rho = \rho_{AB}$, the entanglement cost under (asymptotically) dually non-entangling operations coincides with the corresponding entanglement cost under (asymptotically) non-entangling operations:

$$E_{c,\mathrm{DNE}}(\rho) = E_{c,\mathrm{NE}}(\rho), \tag{13}$$

$$E_{c,\mathrm{ADNE}}(\rho) = E_{c,\mathrm{ANE}}(\rho) = D^{\infty}(\rho\|\mathcal{S}). \tag{14}$$

The last equality in (14) comes from the known result[24] that the ANE entanglement cost of a state ρ is given by the regularised relative entropy of entanglement $D^{\infty}(\rho \,\|\, \mathcal{S})$. It was also recently shown that $E_{c,\mathrm{NE}}(\rho) > D^{\infty}(\rho \,\|\, \mathcal{S})$ in general[25], so a similar equivalence cannot hold for DNE operations without asymptotic entanglement generation.

Theorems 1 and 2 together show that the asymptotic properties of entanglement manipulation under ADNE operations are governed by two entropic quantities: the Piani relative entropy $D^{\mathbb{SEP},\infty}(\cdot\|\mathcal{S})$ in distillation, and the relative entropy $D^{\infty}(\cdot\|\mathcal{S})$ in dilution. This provides a direct motivation for the question: is there actually any difference between the two entropic quantities, or are they simply equal?

Gap with the relative entropy

To demonstrate a gap between $D^{\mathbb{SEP},\infty}(\rho \,\|\, \mathcal{S})$ and $D^{\infty}(\rho \,\|\, \mathcal{S})$, we introduce a simple but general single-letter upper bound for the regularised Piani relative entropy: namely,

$$E_{d,\mathrm{DNE}}(\rho) = D^{\mathbb{SEP},\infty}(\rho \,\|\, \mathcal{S}) \le \widetilde{E}_{\kappa}^{\infty}(\rho) \le \widetilde{E}_{\kappa}(\rho), \tag{15}$$

where

$$\widetilde{E}_{\kappa}(\rho) := \log_2 \min\left\{\operatorname{Tr} S :\ -S \le \rho^{\Gamma} \le S,\ S \in \operatorname{cone}(\mathcal{S})\right\} \tag{16}$$

is a modification of an entanglement monotone E_κ studied in a different context by Wang and Wilde[54]. As before, cone($\mathcal{S}$) denotes here the cone of separable operators. Any ansatz for Eq. (16) then yields an upper bound on $E_{d,\mathrm{DNE}}(\rho)$, allowing one to estimate its value. We now apply the bound of Eq. (15) to the so-called *antisymmetric state* – a special entangled state that has been the subject of many investigations[55–57] in light of its peculiar properties, earning the name of 'universal counterexample' in entanglement theory[49]. It is proportional to the projector onto the antisymmetric subspace within the bipartite Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$, thus being given by $\alpha_d := \frac{1-F}{d(d-1)}$, where $F|\psi\rangle|\phi\rangle := |\phi\rangle|\psi\rangle$ is the swap operator. It is $(d-1)$-extendible on either sub-system, thus it has small distillable entanglement, distillable key, and squashed entanglement – all $O(1/d)$[56]. At the same time, its entanglement cost and regularised relative entropy of entanglement $D^{\infty}(\alpha_d \,\|\, \mathcal{S})$ are larger than a fixed constant independent of d[56]. Our third main result states that the regularised Piani relative entropy of entanglement $D^{\mathbb{SEP},\infty}(\alpha_d \,\|\, \mathcal{S})$ behaves unlike $D^{\infty}(\alpha_d \,\|\, \mathcal{S})$ and instead goes to 0 as $d \to \infty$, again as $O(1/d)$. The existence of a state with this property was left as an open problem by Li and Winter [ref. 32, Fig. 1].

***Theorem* 3.** The antisymmetric state α_d satisfies that

$$D^{\mathbb{SEP},\infty}(\alpha_d \,\|\, \mathcal{S}) \le \log_2\left(1 + \frac{2}{d}\right) \underset{d\to\infty}{\longrightarrow} 0, \tag{17}$$

while $D^{\infty}(\alpha_d \,\|\, \mathcal{S}) \ge \frac{1}{2}\log_2\frac{4}{3} \approx 0.2075$ for all d. Consequently, $D^{\mathbb{SEP},\infty}(\alpha_d \,\|\, \mathcal{S}) < D^{\infty}(\alpha_d \,\|\, \mathcal{S})$ for all $d \ge 13$.

Although this might seem a purely mathematical result, we can give it a direct physical meaning in two contexts: one, in the manipulation of entanglement, and two, in quantum hypothesis testing. We will now explain both of these interpretations in detail.

Irreversibility of entanglement manipulation

Theorems 1, 2, and 3 together imply a fundamental irreversibility of entanglement under asymptotically dually non-entangling operations: it holds that

$$E_{c,\mathrm{ADNE}}(\alpha_d) > E_{d,\mathrm{ADNE}}(\alpha_d) \tag{18}$$

for the antisymmetric Werner state with $d \ge 13$. What this means is that there is no hope of establishing reversibility under DNE transformations, even assisted by asymptotic entanglement generation.

This further hints at the fact that ANE might truly be the smallest class of operations that enable reversibility of entanglement manipulation, as suggested by Plenio[58]. The validity of this conjecture hinges on the generalised quantum Stein's lemma[26,47], arguably one of the main open problems in the field. Our results shed further light on this problem, pinpointing which assumptions are key to reversibility. In a recent work[25], we showed that asymptotic entanglement generation is necessary: allowing only non-entangling operations leads to irreversibility. Here we show that the entanglement generation in itself is not enough: one really needs the full operational power of non-entangling operations *and* asymptotic entanglement generation in order to have any hope of establishing a reversible theory of entanglement. This teaches us something not only about entanglement, but also about the mathematical structure of quantum resource theories: namely, that asymptotically dually resource non-generating operations[59] are in general not enough to unlock reversibility.

Interestingly, the antisymmetric state α_d can actually be shown to be *reversible* under some other classes of free operations that extend LOCC, such as so-called PPT operations[60]. This shows that the more limited distillation power of (A)DNE operations has crucial operational consequences, and (A)DNE is a useful outer approximation to LOCC that is rather independent from other commonly employed ones such as PPT.

We note also that recently a framework was formulated that enables reversibility for all quantum states by allowing for the use probabilistic ANE operations in addition to quantum channels[61]. Although very closely related to the question studied here, this requires a suitable adjustment of the definition of asymptotic rates, and it is not directly comparable with the standard definitions of transformation rates used in this work.

Entanglement testing with separable measurements

Recall from Eq. (11) that the Piani relative entropy $D^{\mathbb{SEP},\infty}(\rho \,\|\, \mathcal{S})$ exactly quantifies the asymptotic performance in hypothesis testing of against all separable states using only separable measurements.

Already in refs. 24,47 an important conjecture was made: that when all measurements are allowed – not just separable ones, but also general global measurements – then the asymptotic rate of entanglement testing equals the regularised relative entropy of entanglement $D^{\infty}(\rho \,\|\, \mathcal{S})$. Because of the connection between entanglement testing and entanglement distillation under NE operations (Eq. (12)), this would also show that $E_{d,\mathrm{NE}}(\rho) = D^{\infty}(\rho \,\|\, \mathcal{S})$ and hence that entanglement is reversible under NE. This conjecture is precisely the generalised quantum Stein's lemma. We have already shown a gap between $D^{\mathrm{SEP},\infty}(\rho \parallel \mathcal{S})$ and $D^{\infty}(\rho \parallel \mathcal{S})$, but without the generalised quantum Stein's lemma, it is impossible to conclude if this gap extends to the operational performance of NE and DNE or to entanglement testing.

Fortunately, we can show that the conjectured generalised quantum Stein's lemma[47] holds true for the state α_d, implying that $E_{d,\mathrm{NE}}(\alpha_d) = D^{\infty}(\alpha_d \parallel \mathcal{S})$ for this state. (See Supplementary Note IV.) Together with Theorems 1 and 3, this then shows that

$$E_{d,\mathrm{DNE}}(\alpha_d) < E_{d,\mathrm{NE}}(\alpha_d) \tag{19}$$

for sufficiently large d. On the one hand, this demonstrates a gap in the operational performance of the two sets of operations in entanglement distillation, showing that DNE provide a strictly tighter approximation to LOCC. On the other hand, it also directly shows a difference in the performance of hypothesis testing in the two settings: in the entanglement testing of the antisymmetric state α_d, global measurements can do *strictly better* than separable (and hence also local) ones. This is the first known asymptotic gap of this kind.

Discussion

We have contributed to the asymptotic theory of entanglement manipulation by exactly computing the rate at which entanglement can be distilled from any quantum state using dually non-entangling operations. In addition to being one of the few examples where asymptotic rates for general quantum states can be evaluated in terms of a single regularised quantity, this result also gives a direct operational interpretation to the Piani relative entropy of entanglement, a known mathematical tool that had not been directly connected with entanglement manipulation before. We provided new insights into the asymptotic properties of the antisymmetric state – a state that has received significant attention due to its often unusual entanglement properties, and whose characterisation is an important problem in entanglement theory – by resolving an open question regarding its distinguishability. Finally, we have shown that dually non-entangling operations assisted by asymptotically vanishing amounts of entanglement can dilute entanglement at a rate given by the regularised relative entropy of entanglement, matching the performance of the larger class of asymptotically non-entangling operations and ruling out the reversibility of entanglement in this setting.

Despite the axiomatic character of DNE operations, our results lead to the establishment of new practically relevant connections – in particular, that between DNE entanglement distillation and entanglement testing with separable measurements – and our precise characterisation of the properties of DNE sheds light on several important physical phenomena in entanglement theory. For instance, through the hypothesis testing connections built and strengthened here, we showed that there is a strict gap in the asymptotic performance of separable vs. global measurements in distinguishing entangled states from unentangled ones. Further, we now know that DNE operations exhibit no bound entanglement, and this immediately implies that a solution to the important open problem of the existence of NPT bound entanglement may only be obtained by looking at classes of operations strictly smaller than DNE. Similarly, our irreversibility result shows that any reversible framework must use operations larger than ADNE. Such insights can be regarded as no-go results that illuminate the extremely complicated and still little-understood questions about the power of different operations in transforming entangled states.

We hope that our results can find use in the understanding of the often enigmatic landscape of asymptotic entanglement manipulation. Characterising the distillability and reversibility properties, as well as evaluating the distillable entanglement for other types of operations remain major open problems in the field.

Table 1 | The distillable entanglement $E_{d,\mathcal{F}}$ and the entanglement cost $E_{c,\mathcal{F}}$ under four different classes of free operations $\mathcal{F}$, namely, non-entangling (NE), asymptotically non-entangling (ANE), dually non-entangling (DNE), and asymptotically dually non-entangling (ADNE) operations

Operations	Distillable entanglement	Entanglement cost	Reversibility?
NE	$D^{\infty}(\cdot \parallel \mathcal{S})$?	⋆	No[25]
ANE	$D^{\infty}(\cdot \parallel \mathcal{S})$?	$D^{\infty}(\cdot \parallel \mathcal{S})$	Yes ?[24,26]
DNE	$D^{\mathrm{SEP},\infty}(\cdot \parallel \mathcal{S})$	⋆	No[25]
ADNE	$D^{\mathrm{SEP},\infty}(\cdot \parallel \mathcal{S})$	$D^{\infty}(\cdot \parallel \mathcal{S})$	No (Thms. 1–3)

The first two rows summarise results in the prior literature, while the last two contain our new findings. The distillable entanglement under NE and ANE is the same, and it is conjectured to coincide with the regularised relative entropy of entanglement $D^{\infty}(\cdot\|\mathcal{S})$[24,26,47,48].
In this work, we completely characterise entanglement manipulation under DNE and ADNE channels, which are less permissive and thus more practically relevant than their NE or ANE counterparts. For both classes, the distillable entanglement is given by the regularised Piani relative entropy $D^{\mathrm{SEP},\infty}(\cdot\|\mathcal{S})$ (Theorem 1). The ADNE entanglement cost is the same as that under ANE, and it thus equals $D^{\infty}(\cdot\|\mathcal{S})$ (Theorem 2). The cost under DNE, instead, coincides with that under the NE class: while it is natural to conjecture that this might also be given by *some* regularised relative entropy expression, we do not yet have any explicit expression for it, and we thus marked it with the symbol ⋆. Our irreversibility result in Theorem 3 then implies that there is a gap between the distillable entanglement and the entanglement cost under (A)DNE operations; furthermore, through a careful analysis of asymptotic hypothesis testing, we can show an operational gap in the power of NE and DNE operations in entanglement distillation.

Note added: At the time of publication, two proofs of the generalised quantum Stein's lemma have been claimed[62,63]. To recall, the validity of this result would establish the conjectured equality between the NE distillable entanglement and the regularised relative entropy of entanglement $D^{\infty}(\cdot\|\mathcal{S})$ for all states, complementing our main results. Our investigation of the antisymmetric state in Supplementary Note IV provides an independent and simpler proof of a special case of this lemma.

We also note that the result of Theorem 3 can be deduced from a claim found in the first pre-print version of ref. 64 (Table 1, 4th row, 3rd column), available at https://arxiv.org/abs/2011.13063v1. No complete proof of this claim was available when the first pre-print of this work appeared, in July 2023. A proof using a different technique than ours can however be found in the published version of ref. 64 (Corollary 14).

Data availability

No data sets were generated or analysed during this study.

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